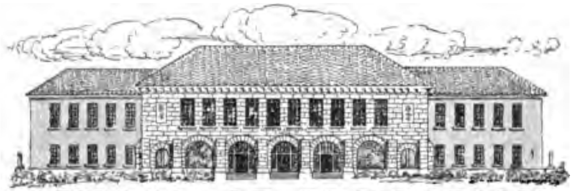
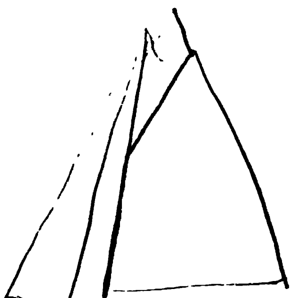
EASY
ALGEBRA





AN EASY ALGEBRA FOR BEGINNERS;

BEING A SIMPLE, PLAIN PRESENTATION OF THE ESSENTIALS OF ELEMENTARY ALGEBRA, AND ALSO ADAPTED TO THE USE OF THOSE WHO CAN TAKE ONLY A BRIEF COURSE IN THIS STUDY.

BY

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PREFACE.

THIS book is designed for the use of those for whom the High School (Elementary) Algebra may be too difficult, and is adapted also to those who can take only a brief course. It has been carefully prepared with a view to render an acquaintance with the essentials of elementary algebra easy of acquisition by the young beginner. The explanations are brief and simple. The examples are not difficult.

Such definitions and rules as are common to arithmetic and algebra have been given without labored illustration, or have been assumed as already familiar to the pupil.

As a preparation for the solution of problems by means of equations, well graded steps are given in the section preceding these problems for practice in the translation of quantitative statements from ordinary language into algebraic expressions.

Only the leading and more easily understood principles of Radicals and Progressions have been introduced, and I trust that these more advanced topics are presented in a simple and attractive form.

The Miscellaneous Examples for independent exercise on the subjects of the different sections will give the pupil greater familiarity with the methods learned in the text; while the Review Questions will serve as a test of the accuracy of his knowledge of the principles underlying these methods and operations.

C. S. V.

University of Virginia,
November 10, 1880.

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EASY ALGEBRA.

SECTION I.

DEFINITIONS.

1. Algebra.—In algebra we use letters to denote numbers, and signs to indicate the arithmetical operations to be performed on them.

2. Signs.—The usual signs, some of which we have used in arithmetic, are :

(1.) The sign of addition + (*plus*), as $a + b$.

(2.) The sign of subtraction − (*minus*), as $a - b$.

(3.) The signs of multiplication $\times$, $\cdot$, and simply writing letters one after another, as $a \times b$, $a \cdot b$, and ab, all mean a multiplied by b.

So, also, $5 \times a$, $5 \cdot a$, and $5a$ mean 5 times a.

NOTE.—In writing figures we must remember that 5·7 means 5 and $\frac{7}{10}$, and 57 means 50 + 7, and not 5 × 7.

(4.) The signs of division $\div$, and a line between the letters. Thus,

$$a \div b, \text{ and } \frac{a}{b} \text{ mean } a \text{ divided by } b.$$

(5.) The sign of equality =, read "is equal to." Thus,

$$\frac{a}{b} = c \text{ is read "} a \text{ divided by } b \text{ is equal to } c.\text{"}$$

3. Coefficient.—As $6 + 6 + 6 + 6 + 6$ is 5×6. So $a + a + a + a + a$ is $5 \times a$, which we write $5a$, and 5 is called the *coefficient* of a. Similarly, $ab + ab + ab + ab + ab + ab = 6ab$, and 6 is the coefficient of the product ab.

Definition.—*The coefficient is the number written before a letter or quantity to show the number of times it is taken.* When no coefficient is written, the coefficient 1 is understood.

4. Algebraic Quantity, or Algebraic Expression.—Any collection of letters with algebraic signs is called an *algebraic quantity,* or an algebraic *expression.* Thus,

$$a,\ a + b + c - d,\ ab,\ 5ab + 2cd - 3ef$$

are algebraic expressions.

5. Terms.—The *terms* of an algebraic expression are the different parts separated by the sign + or −. Thus, in the expressions $a + b + c - d$, $5ab + 2cd - 3ef$, a, b, c, d, $5ab$, $2cd$, and $3ef$ are the terms.

6. Monomial, Polynomial, etc.—An algebraic expression of one term only is called a *monomial;* an expression of two terms is a *binomial;* one of three terms, a *trinomial.* In general, an expression of more than *one* term is called a *polynomial.*

7. Factors.—Just as 5 and 7 are *factors* of 35, so 7 and a are *factors* of $7a$; so a, b, and c are factors of the expression abc.

8. Power and Index, or Exponent.—When the same factor occurs several times, as $a \times a \times a \times a \times a$, or

aaaaa, we write it, for the sake of shortness, a^5. This a^5, thus written, is called the ***fifth power*** of the number a, and is read "a to the fifth power." So, also, $a \times a \times a$ is written a^3, and is read "a to the third power." $a \times a$ is written a^2, and read "a squared," or "a to the second." The 5, 3, and 2 thus written are called ***exponents*** or ***indices***.

Definition.—*An index or exponent of a letter is a small number placed over it to the right to show its power, or how many times it is taken as a factor.* When no exponent is written, the exponent 1 is understood.

EXAMPLES.—Read a^7, b^9, 2^5, 3^4, a^2b^7, b^3c^3, and write them with all their factors.

9. Positive and Negative Quantities.—All terms or quantities which have the *plus* sign, or no sign, before them are *additive*, and are called *positive* quantities.

All quantities with the *minus* sign before them are *subtractive*, and are called *negative* quantities.

10. Like Terms.—Like terms are those which differ only in their numerical coefficients. All others are *unlike*.

Thus, $7a$, $-8a$, and $+5a$ are *like terms*; as, also, $8a^3$ and $-6a^3$; $16a^2b$ and a^2b.

11. Like Signs.—When two quantities are both *plus*, or both *minus*, they are said to have *like* signs. When one is *plus* and the other *minus*, they are said to have *unlike* signs.

NOTE.—The pupil should now be practiced in reading algebraic expressions, and in writing them down from dictation.

SECTION II.

ADDITION.

12. To add like algebraic quantities.

Rule.—*Add separately the plus and minus coefficients, take the difference of the two sums, prefix to this the sign of the greater, and attach the common letter or letters.*

Examples—1.

$$\begin{array}{rrrr}
(1.) & (2.) & (3.) & (4.) \\
2a & 5ax & -\ 6b & -\ 4ab \\
6a & 4ax & -\ 3b & -\ 3ab \\
4a & ax & -\ b & -\ 2ab \\
a & 2ax & -\ 10b & -\ ab \\
\hline
13a & 12ax & -\ 20b & -\ 10ab
\end{array}$$

$$\begin{array}{lrrrrrr}
 & (5.) & (6.) & (7.) & (8.) & (9.) & (10.) \\
\text{To} & 3a & 3a & -3a & -7a & 8a & a \\
\text{add} & -a & 4a & 6a & 5a & -10a & -b \\
\cline{2-7}
 & 2a & 7a & +3a & -2a & -\ 2a & a-b
\end{array}$$

$$\begin{array}{rrrr}
(11.) & (12.) & (13.) & (14.) \\
5x & 10a^2 & 15ab & 12cx^2 \\
-\ 4x & 3a^2 & -\ 10ab & -\ 4cx^2 \\
+\ 2x & -\ 4a^2 & -\ ab & -\ 9cx^2 \\
\hline
+\ 3x & +\ 9a^2 & +\ 4ab & -\ cx^2
\end{array}$$

$$\begin{array}{ccc}
(15.) & (16.) & (17.) \\
5a - 3c & 5a - 3c & 5a + 3c \\
6a - 8c & 6a + 8c & 6a - 8c \\
\hline
11a - 11c & 11a + 5c & 11a - 5c
\end{array}$$

(18.)

$$\begin{array}{r} 6a - 2b + 5c \\ 4a + 8b - 2c \\ \hline 10a + 6b + 3c \end{array}$$

(19.)

$$\begin{array}{l} a - b + c \\ a + b - c \\ \hline 2a \end{array}$$

(20.)

$$\begin{array}{llll} x & - y & + & z \\ x & + y & + & z \\ \hline 2x & & + & 2z \end{array}$$

(21.)

$$\begin{array}{r} 3a - 4b - c \\ 6a + 9b - 7c \\ -5a + 2b - 4c \\ \hline \end{array}$$

(22.)

$$\begin{array}{r} -21a^2 - 14ab + 20ac^2 + 30ac \\ 45a^2 - 20ab - 12ac^2 - 16ac \\ -3a^2 + 2ab + 25ac^2 - 12ac \\ \hline \end{array}$$

Remark.—*Unlike Terms.*—When unlike terms occur, unite them in the sum with their proper signs.

Thus, to add $8c - 5d$ to $3a^2 - b$, we simply write it $3a^2 - b + 8c - 5d$; again, the sum of $3a^2 - 4ab$ and $-2a^2$ is $a^2 - 4ab$.

(23.)

Add $a + 2b - c$ and $a - 6m + 2c$. Thus:

$$\begin{array}{lllll} a & + 2b & - c & & \\ a & & + 2c & - 6m & \\ \hline 2a & + 2b & + c & - 6m & \end{array}$$

(24.)

Add $6x^2 - 8x + a$ and $3x - y + 6$. Thus:

$$\begin{array}{lllll} 6x^2 & - 8x & + a & & \\ & 3x & & - y & + 6 \\ \hline 6x^2 & - 5x & + a & - y & + 6 \end{array}$$

(25.)

Add $3a^2 + 4bx - c^2 + 10$, $-5a^2 + 6ac + 2c^2 - 15a$, $-4a^2 - 3ac - c^2 + 21$.

$$\begin{array}{llllll} 3a^2 & + 4bx & - c^2 & + 10 & & \\ -5a^2 & & + 2c^2 & & + 6ac & - 15a \\ -4a^2 & & - c^2 & + 21 & - 3ac & \\ \hline -6a^2 & + 4bx & & + 31 & + 3ac & - 15a \end{array}$$

Examples—2.

Add together

1. $-6a$, $8a$, $-13a$, $3a$, $-4a$.

2. $x+y+z$, $-x+y+z$, $x-y+z$, $x+y-z$.

3. $3-a$, $-8-a$, $7a-1$, $-a-1$, $9+a$.

4. $b^3-2ab^2+a^2b$, b^3+3ab^2, $2a^3-ab^2-a^2b$.

5. $2x^4-7x^2+3$, $-4x^3+6x^2-2x+7$, x^4-2x^3-4x, $6x^3-9x-12$.

In like manner, by grouping and adding like terms,

6. Reduce the polynomial $a^4-6a^2b+6ab^2-2b^3+5a^4-3ab^2+6a^2b+b^3-4a^4+2a^2b$ to its simplest form.

7. Reduce $4ay^2-3xz+8ab+7xz-6ab+c+8ay^2+4xz+7ab-c+7ay^2-8xz-9ab$ to its simplest form.

SECTION III.

SUBTRACTION.

13. To subtract $+b$ from a, we have $a-b$. To subtract $-b$ from a, we can write for a, $a+b-b$, as it is the same. Now $-b$ taken from $a+b-b$ leaves $a+b$. Hence, $-b$ taken from a gives $a+b$.

In like manner, b subtracted from $-a$ gives $-a-b$, and $-b$ subtracted from $-a$ gives $-a+b$, $-2a$ subtracted from $3a$ gives $+5a$.

Hence,

Rule.—*Change the sign of every term in the subtrahend, and proceed as in addition.*

Examples—3.

1. From $4a$, take a: $3a$

$$\begin{array}{lr} \text{From} & 4a \\ \text{take} & a \\ \hline & 3a \end{array}$$

2. $$\begin{array}{lr} \text{From} & 5x \\ \text{take} & 4x \\ \hline & x \end{array}$$

3. $$\begin{array}{lr} \text{From} & 4a \\ \text{take} & -\ a \\ \hline & 5a \end{array}$$

4. $$\begin{array}{lr} \text{From} & b \\ \text{take} & b \\ \hline & 0 \end{array}$$

5. $$\begin{array}{lr} \text{From} & a \\ \text{take} & -\ a \\ \hline & 2a \end{array}$$

6. $$\begin{array}{lr} \text{From} & 5a \\ \text{take} & -\ 4a \\ \hline & 9a \end{array}$$

7. $$\begin{array}{lr} \text{From} & -\ 4a \\ \text{take} & a \\ \hline & -\ 5a \end{array}$$

8. $$\begin{array}{lr} \text{From} & -\ 5b \\ \text{take} & 4b \\ \hline & -\ 9b \end{array}$$

9. $$\begin{array}{lr} \text{From} & -\ a \\ \text{take} & a \\ \hline & -\ 2a \end{array}$$

10. $$\begin{array}{lr} \text{From} & -\ 4a \\ \text{take} & -\ a \\ \hline & -\ 3a \end{array}$$

11. $$\begin{array}{lr} \text{From} & -\ 5c \\ \text{take} & -\ 4c \\ \hline & -\ c \end{array}$$

12. $$\begin{array}{lr} \text{From} & -\ a \\ \text{take} & -\ a \\ \hline & 0 \end{array}$$

13. $$\begin{array}{lr} \text{From} & x + y \\ \text{take} & x - y \\ \hline & 2y \end{array}$$

14. $$\begin{array}{lr} \text{From} & b - c \\ \text{take} & b + c \\ \hline & -\ 2c \end{array}$$

15. $$\begin{array}{lr} \text{From} & a + bc \\ \text{take} & a - cx \\ \hline & bc + cx \end{array}$$

(16.)

$$\begin{array}{lr} \text{From} & 4a - 5b + 7c \\ \text{take} & a - 3b + 10c - 4x \\ \hline \textit{Ans.} & 3a - 2b - 3c + 4x \end{array}$$

(17.)

$$\begin{array}{lr} \text{From} & 8x - 2y + 4z - 5 \\ \text{take} & 7x - 5y + 4z - 4 \\ \hline \textit{Ans.} & x + 3y \qquad\quad - 1 \end{array}$$

(18.)

$$\begin{array}{lr} \text{From} & a^2 + 3ab - 4c^2 \\ \text{take} & 2a^2 - 6ab - 8c^2 \\ \hline \textit{Ans.} & -\ a^2 + 9ab + 4c^2 \end{array}$$

(19.)

$$\begin{array}{lr} \text{From} & x^3 - 4x^2 + 8x - 11 \\ \text{take} & x^4 - 5x^2 + 10x - 9 \\ \hline \textit{Ans.} & -x^4 + x^3 + x^2 - 2x - 2 \end{array}$$

Examples—4.

1. From a take $a - b - x$.
2. From $2a + 3b - c - d$ take $2a - 3b + c - d$.
3. From $8a - b - c$ take $a - b + 4c$.
4. From $3a + 2x - 5b$ take $2a + 3x + 4b$.
5. From $xy + 2x^2 + 3y^2$ take $xy - 2x^2 + 3y^2$.
6. From $4mn + 5m - 6n$ take $2mn + m + n$.
7. From $3a^2b + 4a^2c - 6c^2$ take $a^2b - a^2c - 8c^2$.
8. From $\frac{2}{3}ab - \frac{1}{3}bc + \frac{4}{3}$ take $\frac{1}{3}ab + \frac{2}{3}bc - \frac{2}{3}$.
9. From $a^3 - 30a^2x + 51ax^2 - 84x^3$ take $a - 35a^2x - 60ax^2 - 250x^3$.

SECTION IV.

MULTIPLICATION.

14. We have seen that a^5 is *aaaaa*, and a^4 is *aaaa*. Therefore, $a^5 \times a^4$ is $aaaaa \times aaaa = a^9$.

Hence, to multiply powers of the same letter, *we add the exponents.*

15. 1. $+3a \times +2b$ means $+3a$ added $2b$ times, or $+6ab$.

2. $-3a \times +2b$ means $-3a$ added $2b$ times, or $-6ab$.

3. $+3a \times -2b$ is the same as $-2b \times 3a$, or $-6ab$, as above.

4. $-3a \times -2b$ means $-3a$ subtracted $2b$ times, that is, $-6ab$ subtracted, which by the rule of subtraction (Art. 13) gives $+6ab$.

Hence, summing up, we have

$$+3a \times +2b = +6ab,$$
$$-3a \times +2b = -6ab,$$
$$+3a \times -2b = -6ab,$$
$$-3a \times -2b = +6ab.$$

Hence, + by + gives + ; − by − gives + ; + by − gives − ; and − by + gives − . Therefore,

16. To multiply a monomial by a monomial,

Rule.—

I. *For Coefficients : Common multiplication.*

II. *For Signs : Like signs make + ; Unlike signs, −.*

III. *For Exponents : Add the exponents of the same letters.*

EXAMPLES :

1. $3a^2 \times 4a^3 = 12a^5$.
2. $5ab \times 4abc = 20a^2b^2c$.
3. $4a^4 \times -axy = -4a^5xy$.
4. $-5x^2y^2z^2 \times -2x^2yz^2 = 10x^4y^3z^4$.
5. $2ab \times -3cy \times -a^3b^2y = 6a^4b^3cy^2$.
6. $axy \times b$; $3ab \times -x$; $-3mn \times am$; $-xy^2 \times -xy^2$.

17. To multiply a polynomial by a monomial or single term.

Rule.—*Multiply every term in the multiplicand by the multiplier.*

Examples—5.

1. $$\begin{array}{l} 5a^2x^4 - 6abx^2 + 3b^2y^4 \\ -2bc^2 \\ \hline -10a^2bc^2x^4 + 12ab^2c^2x^2 - 6b^3c^2y^4. \end{array}$$

2. $\begin{array}{l} 12a - 7b. \\ 9a \\ \hline \end{array}$ 3. $\begin{array}{l} 5b - 8a. \\ -12a \\ \hline \end{array}$ 4. $\begin{array}{l} 10x^2 - 5ax - 3a^2. \\ 4x^2 \\ \hline \end{array}$

Multiply

5. $2ab - 4ac + 6bd$ by $-2x$.
6. $ac + 2bc$ by $3a$.
7. $2ax + 5by - 3cz$ by $-2xy^2$.
8. $3x^8 - 2x^5 + 4x^2$ by $-7x^2$.

18. **To multiply a polynomial by a polynomial.**

Rule.—*Multiply all the terms of the multiplicand by each term of the multiplier. Then add these products.*

Ex. 1.

$$\begin{array}{rr} 3a + 2b & \\ 5a - 4b & \\ \hline 15a^2 + 10ab & \\ - 12ab & - 8b^2 \\ \hline \end{array}$$

Prod. $15a^2 - 2ab - 8b^2$

Ex. 2.

$$\begin{array}{rr} x + 3 & \\ x - 2 & \\ \hline x^2 + 3x & \\ - 2x & - 6 \\ \hline \end{array}$$

Prod. $x^2 + x - 6$

Ex. 3.

$$\begin{array}{rr} a^2 + 2x^2 & \\ 3a^2 + x^2 & \\ \hline 3a^4 + 6a^2x^2 & \\ a^2x^2 & + 2x^4 \\ \hline \end{array}$$

Prod. $3a^4 + 7a^2x^2 + 2x^4$

Ex. 4.

$$\begin{array}{llllll} 2a & + 3b & - 5c & & & \\ a & + b & - c & & & \\ \hline 2a^2 & + 3ab & - 5ac & & & \\ & + 2ab & & + 3b^2 & - 5bc & \\ & & - 2ac & & - 3bc & + 5c^2 \\ \hline 2a^2 & + 5ab & - 7ac & + 3b^2 & - 8bc & + 5c^2. \end{array}$$

Examples—6.

Multiply

1. $a + x$ by $c + y$.
2. $5x + 4$ by $x - 2$.
3. $x - 5$ by $x + 4$.
4. $3x - 4$ by $2x - 3$.
5. $1 - 2x$ by $x - x^2$.
6. $ac - b^2$ by $c^2 - ab$.
7. $-11x - 3a$ by $-10x - 8a$.
8. $1 + 3x + 2y$ by $x - y$.
9. $ab - bc + ac$ by $2a - b$.
10. $x^3 + x^2 + x + 1$ by $x - 1$.

Multiply

11. $5 + 2x + x^2$ by $5 - 2x + x^2$.
12. $x + 4 - y$ by $x + 4 + y$.
13. $3a^4x^2 + 2b^2y$ by $3a^4x^2 - 2b^2y$.
14. $2x^3 + 4x^2 + 8x + 16$ by $x - 2$.
15. $a^4x^4 - a^3x^3y + a^2x^2y^2 - axy^3 + y^4$ by $ax + y$.
16. $a^2 - 2ab + 2b^2$ by $a^2 + 2ab + 2b^2$.
17. $a^2 + b^2 + c^2 - ab - ac - bc$ by $a + b + c$.
18. $1 - 2a + 3a^2 - 4a^3 + 5a^4$ by $1 + 2a + a^2$.
19. $x^2 - 5x - 9$ by $x^2 - 5x + 9$.
20. Find the continued product of $x - 2$ by $x - 2$ by $x - 2$.
21. Multiply $a^2 + 2a - 1$ by $a^2 - a + 1$ and by $a^2 - 3a - 1$, and subtract the second product from the first.
22. Multiply $1 - 2x + 3x^2 - 4x^3$ by $1 - x - x^2$.

SECTION V.

BRACKETS OR PARENTHESIS.

19. Two or more terms are sometimes put in brackets or a parenthesis, and considered as a single term. The sign before the brackets indicates the operation to be performed on all the terms in them. If we remove the brackets, the operation must be performed.

Brackets in Addition and Subtraction.

20. When the + sign is before the brackets, the terms are to be added. Thus, $a + (b + c - d) = a + b + c - d$. When the $-$ sign is before the brackets, the terms are to be subtracted. Thus, $a - (b + c - d) = a - b - c + d$. Therefore we have the

Rule.—*When we remove brackets with the + sign before them, the signs of the terms within remain unchanged. When we remove brackets with the − sign before them, we must change the signs of the terms within.*

Ex. 1. $2a - b + (b - a) = 2a - b + b - a = a$.

Ex. 2. $2 - a - (2 - 2a) = 2 - a - 2 + 2a = a$.

Ex. 3. $3 + a + (-a - 3) = 3 + a - a - 3 = 0$.

Ex. 4. $1 - b + b^2 - (-1 + 2b) - (-b^2 + 1)$

$$= 1 - b + b^2 + 1 - 2b + b^2 - 1 = 1 - 3b + 2b^2.$$

Examples—7.

Simplify

1. $6 - (5 + 3) + (2 - 4) - (3 - 10)$.
2. $a - b - c - (a + b - c)$.

3. $2x - 3y - (2x + 3y)$.
4. $1 - b + b^2 - (1 - b + b^2 - b^3)$.
5. $9a + 12b + c - (a + 7b + c) - (8a + 4b)$.
6. $a - [b - c - (d - c)] = a - b + c + (d - c) = a - b + c + d - c = a - b + d$.
7. $6a - [4b - (2a - b)]$.

Brackets in Multiplication.

21. When brackets are used in multiplication, they mean that the number or letter before or after the brackets is to be multiplied by all the terms in them.

Thus, $a(b - c)$ means a multiplied by $b - c$.
$(a + b + c)x$ means $a + b + c$ multiplied by x.

NOTE.—Instead of using brackets to indicate multiplication, we often use a line over the polynomial as follows: $a \,.\, \overline{b + c}.$
This line is called a *vinculum*.

22. When two pair of brackets are used in multiplication, they mean that all the terms in one pair are to be multiplied by the terms in the other.

Thus, $(a + x)(a + 2x)$ means $a + x$ multiplied by $a + 2x$.

$(a + b)^2$ means $(a + b)(a + b)$, *i. e.*, $a + b$ multiplied by $a + b$; $(5a)^2 = (5a)(5a)$, *i. e.*, $5a \times 5a$.

23. Hence, in multiplication we have the

Rule.—*When we remove brackets, we must first perform the multiplication indicated.*

Thus, $a(b - c) = ab - ac$; $a \times \overline{b + x} = ab + ax$;
$(a + x)(a + 2x) = a(a + 2x) + x(a + 2x) = a^2 + 2ax + ax + 2x^2 = a^2 + 3ax + 2x^2$;
$5(x - 2) - 6(x - 3) = 5x - 10 - 6x + 18 = -x + 8$.

Examples—8.

Simplify

1. $6(x-5)+3(x-4)-5(x-2)$.
2. $a(b-c)-b(a-c)+c(a-b)$.
3. $(x-5)(x+5)-6(x^2-25)$.
4. $(a+b+c)x-(a+b-c)x$.
5. $(3x-2)(3x+2)-(3x)^2$.

SECTION VI.

MULTIPLICATION BY INSPECTION.

24. Some polynomials can be multiplied readily by inspection—that is, without putting the quantities one under the other and proceeding by the regular rule—if we first learn certain forms and rules. The following four cases are of most frequent application.

The Square of the Sum of Two Quantities.

25. $(a+b)^2$. Operation.

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \qquad \\ +ab+b^2 \\ \hline a^2+2ab+b^2 \end{array}$$

or $$(a+b)^2=a^2+2ab+b^2 \quad . \quad . \quad .. \quad . \quad (A).$$

This is called a *formula*, and expresses in algebraic language the following

Rule.—*The square of the sum of two quantities is the square of the first, plus twice the product of the first by the second, plus the square of the second.*

Ex. 1. $(x + 5)^2 = x^2 + 10x + 25$.

Ex. 2. $(3a + 2b)^2 = (3a)^2 + 2 \times 3a \times 2b + (2b)^2 = 9a^2 + 12ab + 4b^2$.

The Square of the Difference of Two Quantities.

26. $(a - b)^2$. Operation.

$$\begin{array}{r} a - b \\ a - b \\ \hline a^2 - ab \qquad \\ - ab + b^2 \\ \hline a^2 - 2ab + b^2 \end{array}$$

or $$(a - b)^2 = a^2 - 2ab + b^2. \quad . \quad . \quad . \quad \text{(B)},$$

which expresses the **Rule** :—*The square of the difference of two quantities is the square of the first, minus twice the product of the first by the second, plus the square of the second.*

Ex. 1. $(x - 5)^2 = x^2 - 10x + 25$.

Ex. 2. $(3a - 2b)^2 = (3a)^2 - 2 \times 3a \times 2b + (2b)^2 = 9a^2 - 12ab + 4b^2$.

The Sum of Two Quantities Multiplied by their Difference.

27. $(a + b)(a - b)$. Operation.

$$\begin{array}{r} a + b \qquad \\ a - b \qquad \\ \hline a^2 + ab \qquad \\ - ab - b^2 \\ \hline a^2 \qquad\quad - b^2 \end{array}$$

or $$(a + b)(a - b) = a^2 - b^2 \quad . \quad . \quad . \quad . \text{(C)},$$

which expresses the **Rule** :—*The sum of two quantities multiplied by their difference is the square of the first minus the square of the second.*

Ex. 1. $(x + 5)(x - 5) = x^2 - 5^2 = x^2 - 25.$

Ex. 2. $(2a + 3b)(2a - 3b) = (2a)^2 - (3b)^2 = 4a^2 - 9b^2.$

Ex. 3. $(4b + 3)(4b - 3) = (4b)^2 - 3^2 = 16b^2 - 9.$

The Product of x + *or* – *one Number by* x + *or* – *another Number.*

28. $(x + a)(x + b)$.

Operation.
$$\begin{array}{r} x + a \\ x + b \\ \hline x^2 + ax \qquad \\ + bx + ab \\ \hline x^2 + (a + b)x + ab \end{array}$$

or

$$\left.\begin{array}{l} (x + a)(x + b) = x^2 + (a + b)x + ab \\ \text{Also,} \\ (x - a)(x - b) = x^2 - (a + b)x + ab \\ (x + a)(x - b) = x^2 + (a - b)x - ab \end{array}\right\} \quad . \quad . \quad \text{(D)}.$$

Hence, the **Rule**:—*The product of* x *plus or minus a number, by* x *plus or minus another number, is* x^2 *plus or minus the (algebraic) sum of the numbers multiplied by* x, *plus or minus the product of the numbers, according to the sign.*

Ex. 1. $(x + 2)(x + 3) = x^2 + (2 + 3)x + 2 \times 3 = x^2 + 5x + 6.$

Ex. 2. $(x - 2)(x - 3) = x^2 - (2 + 3)x + (-2 \times -3) = x^2 - 5x + 6.$

Ex. 3. $(x + 3)(x - 2) = x^2 + (3 - 2)x + 3 \times -2 = x^2 + x - 6.$

Ex. 4. $(x - 3)(x + 2) = x^2 + (2 - 3)x + 2 \times - 3$
$= x^2 - x - 6.$

Examples—9.

Write the squares of

1. $m + n$; $m - n$.
2. $m - 2n$; $3a + 4b$.
3. $4a - b$; $5a - 3b$.
4. $7x - 4y$; $a^2 + 3ax$.
5. $1 + 2a^2$; $3x^2 - 2a^2$.

Write the products of

6. $(a + 2b)(a - 2b)$; $(2a + b)(2a - b)$.
7. $(x + 3y)(x - 3y)$; $(3a + 4c)(3a - 4c)$.
8. $(8x + 9y)(8x - 9y)$; $(x^2 + y^2)(x^2 - y^2)$.
9. $(3ax + b)(3ax - b)$; $(mx + 2ay)(mx - 2ay)$.
10. $(4x^2 + 1)(4x^2 - 1)$; $(x^2 - 4)(x^2 + 4)$.

Write the products of

11. $(x + 5)(x + 3)$; $(x - 6)(x - 4)$.
12. $(x - 6)(x + 2)$; $(x - 8)(x + 10)$.
13. $(x + 7)(x - 6)$; $(a + b)(a - c)$.

SECTION VII.

DIVISION.

29. $+ a \times + b = + ab$; hence $+ ab \div + a = + b$.
$+ a \times - b = - ab$; hence $- ab \div + a = - b$.
$- a \times + b = - ab$; hence $- ab \div - a = + b$.
$- a \times - b = + ab$; hence $+ ab \div - a = - b$.

Hence, for signs in division, we have the

Rule.—*Like signs give +, and unlike signs give −.*

30. $a^4 \times a^2 = a^6$; hence, $\dfrac{a^6}{a^4} = a^2$.

Therefore, to divide two powers of the same letter, we *subtract the exponent of the divisor from that of the dividend.*

31. Hence, to divide one monomial, or single term, by another, we have the

Rule.—*Divide the coefficients, observing the rule of the signs. To this quotient annex the letters, subtracting the exponents of the like letters.*

Ex. 1. $-6a^4b^2 \div 3a^2b = -2a^2b.$

Ex. 2. $8a^2b \div 4ab = 2a.$

Ex. 3. $-10a^2bc \div 5a = -2abc.$

Note.—It is plain that if the coefficient of the dividend is not exactly divisible by the coefficient of the divisor, or if a letter enters the divisor which is not in the dividend, or if a letter in the divisor has an exponent greater than the exponent of the same letter in the dividend, the division is impossible, since in all these cases the divisor has factors not contained in the dividend. In such case, the operation is expressed after the manner of a fraction in arithmetic.

Examples—10.

Divide

1. $8x$ by 8; $8a$ by $-a$; abc by $-a$; $-axy$ by y.
2. $12xy$ by $3x$; $-8axy$ by $+8ax$.
3. $-15ab^2c$ by $-3ab$; $-20a^2b^3c^4$ by $5abc$.
4. $-4a^4b^3c^5$ by $-a^2bc^2$; $14a^2xy$ by $7ay$.

32. To divide a polynomial by a monomial.

Rule.—*Divide every term in the dividend by the divisor.*

Ex. 1. $8x\,)\,\underline{40x^2 - 24ax}\,;$

Quotient, $5x - 3a$

Ex. 2. $3x\,)\,\underline{12x^3 - 21ax^2 + 3a^2x}$

Quotient, $4x^2 - 7ax + a^2$

Examples—11.

Divide

1. $2ab + 3ac - 4ad$ by a.
2. $ax + bx^2 - cxy$ by $-x$.
3. $6a^2x^2 - 8abx + 2ax^4$ by $-2ax$.
4. $5bc + 35abc^2 - 10b^2c^2$ by $5bc$.

Division of Polynominals.

33. Definition.—The terms of a polynomial are said to be arranged according to a given letter, when beginning with the highest power of that letter we go regularly down to the lowest, or when we begin with the lowest and go up to the highest.

Thus, $4x^5 - 3x^4 + 2x^2 - 1$ is arranged with reference to x.

34. To divide one polynomial by another.

Rule.—1. *Set down the divisor and dividend as in long division in arithmetic, taking care to arrange them both according to the same letter.*

2. *Divide the first term in the dividend by the first term of the divisor, for the first term of the quotient. Then*

multiply the divisor by the first term of the quotient, and subtract the product from the dividend. Divide the first term of the remainder by the first term of the divisor, and proceed as before, continuing the process with the terms that remain.

Ex. 1. Divide $x^2 + 4ax + 3a^2$ by $x + 3a$.

$$
\begin{array}{r|l}
x + 3a\,) & x^2 + 4ax + 3a^2\,(x + a, \text{ quotient.} \\
 & \underline{x^2 + 3ax} \\
 & \phantom{x^2+{}} ax + 3a^2 \\
 & \phantom{x^2+{}} \underline{ax + 3a^2} \\
 & \phantom{x^2+ax+{}} 0
\end{array}
$$

Ex. 2.

$$
\begin{array}{r|l}
a - b\,) & a^2 - 2ab + b^2\,(a - b, \text{ quotient.} \\
 & \underline{a^2 - ab} \\
 & - ab + b^2 \\
 & \underline{- ab + b^2} \\
 & \phantom{a^2-ab+{}} 0
\end{array}
$$

Ex. 3.

$$
\begin{array}{r|l}
7x-3\,) & 7x^3-24x^2+58x-21\,(x^2-3x+7, \text{ quotient.} \\
 & \underline{7x^3-3x^2} \\
 & -21x^2+58x \\
 & \underline{-21x^2+9x} \\
 & +49x-21 \\
 & \underline{+49x-21} \\
 & 0
\end{array}
$$

Examples—12.

Divide

1. $x^2 + 5x + 6$ by $x + 3$.
2. $8a^2 + 17a + 9$ by $8a + 9$.

3. $21a^2 + ax - 2x^2$ by $7a - 2x$.
4. $x^2 - 13x + 40$ by $x - 5$.
5. $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz$ by $x + y + z$.
6. $a^2 - b^2 - c^2 - 2bc$ by $a - b - c$.
7. $a^2 - a - 30$ by $a + 5$.
8. $a^4 + a^2b^2 + b^4$ by $a^2 + ab + b^2$.
9. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 - 2ab + b^2$.
10. $x^4 + 7x^2 + 2x + 15$ by $x^2 - x + 5$.
11. $xy + 2x^2 - 3y^2 - 4yz - xz - z^2$ by $2x + 3y + z$.
12. $15a^4 + 10a^3x + 4a^2x^2 + 6ax^3 - 3x^4$ by $3a^2 - x^2 + 2ax$.
13. $ab + 2a^2 - 3b^2 - 4bc - ac - c^2$ by $2a + 3b + c$.
14. $3x^4 + 14x^3 + 9x + 2$ by $x^2 + 5x + 1$.
15. $3x^3 + 4abx^2 - 6a^2b^2x - 4a^3b^3$ by $2ab + x$.

35. Remark.—*Incomplete Polynomials.*—Sometimes some of the powers of the letter are wanting in the dividend or divisor, or in both. In such case it is best for beginners to leave a space for the wanting terms.

Ex. 16. Divide $a^5 - 32$ by $a - 2$.

$$
\begin{array}{r|llllll}
a - 2 & a^5 & & & & - 32 & (a^4 + 2a^3 + 4a^2 + 8a + 16. \\
 & a^5 - 2a^4 & & & & & \\
\hline
 & & + 2a^4 & & & - 32 & \\
 & & + 2a^4 - 4a^3 & & & & \\
\hline
 & & & + 4a^3 & & - 32 & \\
 & & & + 4a^3 - 8a^2 & & & \\
\hline
 & & & & + 8a^2 & - 32 & \\
 & & & & + 8a^2 - 16a & & \\
\hline
 & & & & & + 16a - 32 & \\
 & & & & & + 16a - 32 & \\
\hline
 & & & & & 0 &
\end{array}
$$

Ex. 17. Divide $6x^4 - 96$ by $2x^3 + 4x^2 + 8x + 16$.

$$\begin{array}{r|l}
2x^3 + 4x^2 + 8x + 16 & 6x^4 \qquad\qquad\qquad\qquad\quad - 96\ (3x - 6 \\
 & \underline{6x^4 + 12x^3 + 24x^2 + 48x\qquad\ } \\
 & \qquad - 12x^3 - 24x^2 - 48x - 96 \\
 & \qquad \underline{- 12x^3 - 24x^2 - 48x - 96} \\
 & \qquad\qquad\qquad\quad 0
\end{array}$$

Divide

18. $9a^2b^2 - 16x^2$ by $3ab + 4x$.
19. $8x^3 - 27a^3$ by $2x - 3a$.
20. $16a^4 - 81$ by $2a - 3$.

SECTION VIII.

FACTORING.

36. By reversing the formulas in Section VI., we may determine, by inspection, the factors of certain polynomials.

We have seen that $(x + a)(x - a) = x^2 - a^2$. (Art. 27.)

Hence, the factors of $x^2 - a^2$ are $x + a$ and $x - a$. Thus, also, the factors of $x^2 - 25$ are $x + 5$ and $x - 5$; of $9x^2 - 4$ are $3x + 2$ and $3x - 2$.

37. $(x + a)^2 = x^2 + 2ax + a^2$; hence, the factors of $x^2 + 2ax + a^2$ are $x + a$ and $x + a$; of $x^2 + 10x + 25$ are $x + 5$ and $x + 5$. (Art. 25.)

38. $(x - a)^2 = x^2 - 2ax + a^2$; hence, the factors of $x^2 - 2ax + a^2$ are $x - a$ and $x - a$. (Art. 26.)

Ex. The factors of $x^2 - 8x + 16$ are $x - 4$ and $x - 4$.

$(x+4)(x+5)$

39. We have seen $(x + a)(x + b) = x^2 + (a + b)x + ab$; hence, the factors of $x^2 + (a + b)x + ab$ are $x + a$ and $x + b$. (Art. 28.)

Ex. The factors of $x^2 + 5x + 6$ are $x + 3$ and $x + 2$.

40. $(x - a)(x - b) = x^2 - (a + b)x + ab$; hence, the factors of $x^2 - (a + b)x + ab$ are $x - a$ and $x - b$.

Ex. The factors of $x^2 - 7x + 10$ are $x - 5$ and $x - 2$.

41. $(x + a)(x - b) = x^2 + (a - b)x - ab$; hence, the factors of $x^2 + (a - b)x - ab$ are $x + a$ and $x - b$.

Ex. 1. The factors of $x^2 + 5x - 6$ are $x + 6$ and $x - 1$.

Ex. 2. The factors of $x^2 - 5x - 6$ are $x - 6$ and $x + 1$.

Examples—13.

Find the factors of

1. $x^2 + 4x + 4$; $x^2 - 12x + 36$.
2. $4x^2 - 12x + 9$; $a^2 + 2ac + c^2$.
3. $x^2 - 4$; $4a^2 - 4ac + c^2$.
4. $4x^2 - 9$; $16a^2b^2 - 9c^2$.
5. $16a^2 - 64$; $9m^2n^2 - 25$.
6. $x^4 - a^4$; $36a^2x^2 - 25b^2y^2$; $1 - 4x^2$.
7. $x^2 + 9x + 20$; $x^2 + 4x + 3$.
8. $x^2 - 6x + 8$; $x^2 + x - 6$.
9. $x^2 - x - 6$; $x^2 - 8x - 20$.
10. $x^2 - 6x - 7$; $x^2 + 6x - 7$.
11. $x^2 - x - 72$; $x^2 - 2x - 99$.
12. $x^2 - 12ax + 32a^2$; $a^2x^2 - 4abx + 4b^2$.
13. $x^2 - cx - 110c^2$; $x^2 - 21cx + 110c^2$.

SECTION IX.

GREATEST COMMON DIVISOR.

42. To find the greatest common divisor of two monomials.

Rule.—*Find the G. C. D. of the coefficients by the rule in arithmetic, and of the letters separately by inspection, and multiply the results.*

Ex. 1. Find the G. C. D. of $18a^5b^2c$ and $45a^4b^3c^4$.

The G. C. D. of 18 and 45 is 9 ; of a^4 and a^5 is a^4 ; of b^2 and b^3 is b^2 ; of c and c^4 is c.

Hence, the required G. C. D. is $9a^4b^2c$.

Ex. 2. The G. C. D. of $45a^3x^2y$ and $60x^3y^2$ is $15x^2y$.

Ex. 3. The G. C. D. of $54b^2c^3x^3$ and $72b^3c^4x^2$ is $18b^2c^3x^2$.

43. To find the greatest common divisor of two polynomials.

Rule.—*Proceed by the rule for numbers in arithmetic : Divide the greater polynomial by the less, then divide the less by the remainder, and then the first remainder by the second remainder, and so on till there be no remainder. The last divisor will be the Greatest Common Divisor.*

NOTE.—We may take out a numerical factor from any divisor, or multiply any dividend by a numerical factor, if necessary, to make the division possible. Any factor common to all the terms of both polynomials is a part of the G. C. D.

Ex. 1. Find the G. C. D. of $6x^2 - 11x + 4$ and $2x^2 - 5x + 2$.

$$\begin{array}{rl}
2x^2 - 5x + 2\,) & 6x^2 - 11x + 4\,(3 \\
 & \underline{6x^2 - 15x + 6} \\
 & \qquad 4x - 2 \\
\text{or} & \quad 2(2x - 1).
\end{array}$$

$$\begin{array}{rl}
2x - 1\,) & 2x^2 - 5x + 2\,(x - 2 \\
 & \underline{2x^2 - x} \\
 & \quad - 4x + 2 \\
 & \quad \underline{- 4x + 2} \\
 & \qquad 0
\end{array}$$

Hence, $2x - 1$ is the G. C. D.

Ex. 2. Find the G. C. D. of $8x^2 + 7x - 1$ and $6x^2 + 7x + 1$.

$$\begin{array}{rl}
 & 6x^2 + 7x + 1 \\
 & \underline{4 \qquad\qquad\quad} \\
8x^2 + 7x - 1\,) & 24x^2 + 28x + 4\,(3 \\
 & \underline{24x^2 + 21x - 3} \\
 & \qquad 7x + 7 \\
\text{or} & \qquad 7(x + 1).
\end{array}$$

$$\begin{array}{rl}
x + 1\,) & 8x^2 + 7x - 1\,(8x - 1 \\
 & \underline{8x^2 + 8x} \\
 & \quad - x - 1 \\
 & \quad \underline{- x - 1} \\
 & \qquad 0
\end{array}$$

Hence, $x + 1$ is the G. C. D.

Examples—14.

Find the G. C. D.

1. Of $64a^5b^3c^2$ and $48a^4bx$.
2. Of $78a^2x^3$ and $52ax^4$.

3. Of $96c^5x^2y^3$ and $108c^6y^2$.
4. Of $x^2 + 2x + 1$ and $x^2 - 5x - 6$.
5. Of $x^2 - 4x + 4$ and $x^2 - 5x + 6$.
6. Of $x^2 + 8x - 9$ and $x^2 + 17x + 72$.
7. Of $x^2 - 2x + 1$ and $3x^2 - 5x + 2$.
8. Of $3x^2 - 10x + 8$ and $6x^2 - 5x - 4$.
9. Of $3x^2 - 20x + 32$ and $15x^2 - 64x + 16$.

SECTION X.

LEAST COMMON MULTIPLE.

44. To find the least common multiple of two or more monomials.

Rule.—*Find the L. C. M. of the numerical coefficients by the rule in arithmetic, and of the letters separately by inspection, and multiply the results.*

Ex. 1. Find the L. C. M. of $4a^2x^2$ and $6ax^3$.

$$\begin{array}{r|l} 2 & 4,\ 6 \\ \hline & 2,\ 3 \end{array} \qquad \begin{array}{r|l} a & ax^3,\ a^2x^2 \\ \hline x & x^3,\ ax^2 \\ \hline x & x^2,\ ax \\ \hline & x,\ a \end{array}$$

Result, 12.

Result, $axxxa = a^2x^3$.

Hence, L. C. M. $= 12a^2x^3$.

NOTE.—The rule in arithmetic applies to polynomials also.

Ex. 2. Find the L. C. M. of $x^2 - 4$ and $x^2 - 2x + 1$.

$$
\begin{array}{rl}
x - 2\,) & x^2 - 4 \qquad x^2 - 2x + 1 \\
\hline
 & x \;+ 2 \qquad\quad x \;- 2
\end{array}
$$

L. C. M. $= (x - 2)(x + 2)(x - 2) = (x - 2)(x^2 - 4) = x^3 - 2x^2 - 4x + 8$.

Examples—15.

Find the L. C. M. of

1. $2a$, $12ab$, and $8ab$.
2. a^2, b^2, and $2bc$.
3. $16a^2$, $12a^3$, and $30a^4$.
4. ab, ac^2, a^2c, and bc.
5. $8x^4$, $10x^3y$, and $12x^2y^2$.
6. $7a^5$, $42a^2$, and $63a^6$.
7. $18ab^3$ and $12a^3b$.
8. $ax + ay$ and $ax - ay$.
9. $2(a + b)$ and $6(a^2 - b^2)$.
10. $x^2 - 2x + 1$ and $x^2 - 3x + 2$.

SECTION XI.

REDUCTION OF FRACTIONS TO LOWEST TERMS.

45. To reduce a fraction to its lowest terms.

Rule.—*The same as in arithmetic:—Divide the numerator and denominator of the fraction by their greatest common divisor.*

Ex. 1. Reduce $\dfrac{20a^2x^2}{15a^2x^4}$ to its lowest terms.

Cancelling out like factors, we have

$$\frac{\overset{4}{\cancel{20}}\cancel{a^2}\cancel{x^2}}{\underset{3}{\cancel{15}}\cancel{a^2}\underset{x^2}{\cancel{x^4}}} = \frac{4}{3x^2}.$$

Ex. 2. Reduce $\dfrac{2ab - b^2}{3ab - 2b^2}$ to its lowest terms.

$$\frac{2ab - b^2}{3ab - 2b^2} = \frac{(2a - b)\,\cancel{b}}{(3a - 2b)\,\cancel{b}} = \frac{2a - b}{3a - 2b}.$$

Ex. 3. Reduce $\dfrac{x^2 - 9}{x^2 + 6x + 9}$.

$$\frac{x^2 - 9}{x^2 + 6x + 9} = \frac{(x - 3)(x + 3)}{(x + 3)^2} = \frac{x - 3}{x + 3}.$$

Examples—16.

Reduce to their lowest terms

1. $\dfrac{ax}{bx}$, $\dfrac{abc}{bcd}$.

2. $\dfrac{3a - 3b}{3b}$, $\dfrac{a^2 - ab}{ab}$.

3. $\dfrac{25ab^2c^3}{15a^2c^4}$, $\dfrac{abc^3}{2ab^2c^4}$.

4. $\dfrac{x^2 + xy}{x^2 - xy}$, $\dfrac{4a^2b - 5ab^2}{20abc}$.

5. $\dfrac{x^2 - y^2}{x^2 + xy}, \quad \dfrac{x^2 - 2x + 1}{x^2 - 3x + 2}.$

6. $\dfrac{x^2 - a^2}{x^2 + 2ax + a^2}, \quad \dfrac{x^2 - 4}{x^2 + 4x + 4}.$

SECTION XII.

REDUCTION OF FRACTIONS TO A COMMON DENOMINATOR.

46. To reduce fractions to a common denominator.

Rule.—*The same as in arithmetic:—Multiply each numerator by all the denominators, except its own, for new numerators, and the denominators together for the common denominator.* Or,

Multiply each numerator by the quotient resulting from dividing the least common multiple of the denominators by its denominator, and write the L. C. M. of the denominators for the common denominator.

Ex. 1. Reduce $\dfrac{x}{4}, \dfrac{x}{3}, \dfrac{x}{8}$ to a common denominator.

The least common denominator is 24. Hence the result,

$$\frac{6x}{24}, \ \frac{8x}{24}, \ \frac{3x}{24}.$$

Ex. 2. Reduce $\dfrac{ac}{6x^2}, \dfrac{bc}{9x^3}$, and $\dfrac{ab}{36x}$ to a common denominator.

$36x^3$ is the L. C. M. of the denominators. Hence the result,

$$\frac{6acx}{36x^3},\quad \frac{4bc}{36x^3},\quad \frac{abx^2}{36x^3}.$$

Examples—17.

Reduce to a common denominator

1. $\dfrac{x}{3}$, $\dfrac{x}{5}$, and $\dfrac{x}{15}$.

2. $\dfrac{ab^2}{4cx}$, $\dfrac{b+a}{12x^2}$, and $\dfrac{a^2}{16cx}$.

3. a and $\dfrac{2a^2}{a-x}$.

4. $\dfrac{1}{x}$, $\dfrac{1}{2x}$, and $\dfrac{1}{3x}$.

5. $\dfrac{5x+4}{9}$ and $\dfrac{10x+17}{18}$.

6. $\dfrac{5x-1}{4a}$ and $\dfrac{x+2}{26a^2}$.

7. $\dfrac{6a}{x-4}$ and $\dfrac{5a}{x-3}$.

8. $\dfrac{4a}{x-2}$, $\dfrac{3a}{x+2}$, and $\dfrac{6a}{x^2-4}$.

9. $\dfrac{5}{x}$, $\dfrac{6x}{x-1}$, $\dfrac{4}{x+1}$.

SECTION XIII.

ADDITION OF FRACTIONS.

47. To add fractions.

Rule.—*The same as in arithmetic:—Reduce the fractions, if necessary, to a common denominator, then add the numerators and write their sum over the common denominator.*

Ex. 1. Add $\frac{x}{3}$, $\frac{x}{4}$, and $\frac{x}{5}$.

The L. C. M. of the denominators is 60.

Hence, we have $\frac{20x}{60} + \frac{15x}{60} + \frac{12x}{60} =$

$$\frac{20x + 15x + 12x}{60} = \frac{47x}{60}.$$

Ex. 2. Simplify $\frac{5}{x-1} + \frac{10}{x+1}$.

The common denominator is $x^2 - 1$.

Hence, we have $\frac{5(x+1)}{x^2-1} + \frac{10(x-1)}{x^2-1} =$

$$\frac{5x + 5 + 10x - 10}{x^2 - 1} = \frac{15x - 5}{x^2 - 1}.$$

Ex. 3. Simplify $\frac{3a}{4} + \frac{5b}{6} + 2b$.

The L. C. M. of the denominators is 12.

Hence, we have $\frac{9a}{12} + \frac{10b}{12} + \frac{24b}{12} =$

$$\frac{9a + 10b + 24b}{12} = \frac{9a + 34b}{12}.$$

Examples—18.

Add

1. $\dfrac{x}{a}$, $\dfrac{y}{a}$, and $\dfrac{z}{a}$.

2. $\dfrac{a}{x}$ and $\dfrac{b}{3x}$.

3. $\dfrac{x}{2}$, $\dfrac{x}{9}$, and $\dfrac{x}{6}$.

4. $\dfrac{1}{2a}$, $\dfrac{1}{3a}$, and $\dfrac{1}{4a}$.

5. $\dfrac{5x}{6}$, $\dfrac{3x}{5}$, and $\dfrac{7x}{30}$.

6. $\dfrac{a}{3c}$, $\dfrac{a}{6bc}$, and $\dfrac{a}{8bc}$.

7. $\dfrac{a+b}{2}$ and $\dfrac{a-b}{4}$.

8. $\dfrac{3x+1}{8}$ and $\dfrac{4x-5}{24}$.

9. x, $\dfrac{4x-5}{2}$, and $\dfrac{3x-4}{3}$.

10. $2a$ and $\dfrac{b-6a}{3}$.

11. a and $\dfrac{b}{c}$.

12. x and $\dfrac{4-3x}{3}$.

13. $\dfrac{a-x}{ax}$, $\dfrac{x-y}{xy}$, and $\dfrac{y-a}{ay}$.

14. Simplify $\dfrac{5}{x-1}+\dfrac{10}{x+1}+\dfrac{20}{x^2-1}$.

15. Simplify $8+\dfrac{3a-4}{3}$.

16. Simplify $\dfrac{4+3a}{3}+\dfrac{3-4a}{4}$.

SECTION XIV.

SUBTRACTION OF FRACTIONS.

48. To subtract fractions.

Rule.—*The same as in arithmetic:—Reduce the fractions, if necessary, to a common denominator. Subtract the numerator of the subtrahend from that of the minuend and place the result over the common denominator.*

Ex. 1. From $\dfrac{3x}{5}$ take $\dfrac{7x}{12}$.

The common denominator is 60.

Hence, $\dfrac{3x}{5}-\dfrac{7x}{12}=\dfrac{36x}{60}-\dfrac{35x}{60}=\dfrac{36x-35x}{60}=\dfrac{x}{60}$.

Ex. 2. From a take $\dfrac{4a-b}{4}$.

We have

$$\frac{4a}{4}-\frac{4a-b}{4}=\frac{4a-(4a-b)}{4}=\frac{4a-4a+b}{4}=\frac{b}{4}.$$

Note.—The $-$ before $\dfrac{4a-b}{4}$ belongs to the whole numerator, and hence $4a-b$ must be put in brackets with $-$ before it when the common denominator is written.

Examples—19.

1. From $\dfrac{5x}{7}$ take $\dfrac{9x}{14}$.

2. From x take $\dfrac{8x}{9}$.

3. From x take $\dfrac{3x-5}{6}$.

4. From $\dfrac{8x+9}{10}$ take $\dfrac{4x+3}{5}$.

5. From $\dfrac{a+c}{b}$ take $\dfrac{a-c}{b}$.

6. From $\dfrac{c}{a+b}$ take $\dfrac{c}{a-b}$.

7. From $\dfrac{2}{a}+\dfrac{5}{a}$ take $\dfrac{3}{a}+\dfrac{4}{a}$.

8. From $\dfrac{2a}{x+1}$ take $\dfrac{4a}{x+2}$.

9. From $\dfrac{10}{x-2}$ take $\dfrac{5}{x^2-4}$.

10. From $\dfrac{x}{x+y}$ take $\dfrac{x+y}{x}$.

11. From $\dfrac{a+b}{a-b}$ take $\dfrac{a-b}{a+b}$.

12. From $\dfrac{x}{5}+\dfrac{4}{25}$ take $\dfrac{7x-6}{25}$.

13. Simplify $ac-\dfrac{b+c}{a}$.

14. Simplify $\dfrac{4}{x+1}+\dfrac{3}{x-1}-\dfrac{7x-1}{x^2-1}$.

SECTION XV.

MULTIPLICATION OF FRACTIONS.

49. To multiply fractions.

Rule.—*The same as in the arithmetic :—Multiply the numerators together and the denominators together, after cancelling out factors which are the same in the numerators and denominators.*

Ex. 1. $\dfrac{a}{b}\times 5=\dfrac{5a}{b}$.

Ex. 2. $\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} = \frac{ac}{bd}$.

Examples—20.

1. Multiply $\frac{a-b}{a+b}$ by 10.

2. Multiply $\frac{3a-b}{ac}$ by $2a$.

3. Multiply $\frac{3x-5}{16}$ by 96.

4. Multiply $\frac{3a}{2}$ by $\frac{5a}{3}$

5. Multiply $\frac{3-4x}{5}$ by $\frac{3}{4}$.

6. Multiply $\frac{3a^2b}{7cx}$ by $\frac{5ax^2}{a^2b^3}$.

7. Find the continued product of

$$\frac{ma^2}{6b^2} \times \frac{27mb^2}{5a^3} \times \frac{5a}{9m^2b}.$$

8. Multiply $1 + \frac{1}{x}$ by $1 - \frac{1}{x}$.

9. Multiply $\frac{2}{1+x} + \frac{2}{1-x}$ by $\frac{1}{4}$.

10. Multiply $1 - \dfrac{a}{a+1}$ by $1 + \dfrac{a}{1-a}$.

11. Multiply $\dfrac{x^2 - 2x + 4}{x^2 - 8x + 15}$ by $\dfrac{x^2 - 7x + 10}{x^2 - 3x + 2}$.

SECTION XVI.

DIVISION OF FRACTIONS.

50. To divide fractions.

Rule.—*The same as in arithmetic:—Invert the divisor and proceed as in multiplication, cancelling when possible.*

Examples—21.

1. Divide $\dfrac{abx}{cy}$ by a.

2. Divide $\dfrac{ax}{by}$ by b.

3. Divide $\dfrac{3x}{4}$ by 7.

4. Divide $\dfrac{3x}{4}$ by bx.

5. Divide $5ab$ by $\dfrac{3a}{5m}$.

6. Divide $\frac{2a - 6ac}{c}$ by $2a$.

7. Divide $\frac{3a^2bc}{x}$ by $-\frac{ac}{bx}$.

8. Divide $-\frac{a^2x^3y^3}{2bc}$ by $-\frac{ay^2}{4x^2}$

9. Divide $2a + \frac{a}{x}$ by $a - \frac{a}{x}$.

10. Divide $\frac{10(x+y)}{3(x-y)}$ by $\frac{2(x+y)}{9(x-y)}$.

11. Divide $\frac{a^2 - b^2}{ab}$ by $\frac{a+b}{b}$.

12. Divide $\frac{x^2 + 4x + 3}{x^2 - 5x + 6}$ by $\frac{x^2 + 3x + 2}{x^2 - 9}$.

SECTION XVII.

FINDING NUMERICAL VALUES BY SUBSTITUTION.

51. We find a numerical value for an algebraic expression by substituting numbers for the letters in the expression and performing the operations indicated by the signs.

Ex. 1. If $a = 6$ and $b = 2$, then $a - b = 6 - 2 = 4$; $a^2 - b^2 = 36 - 4 = 32$.

Ex. 2. If $a = 2$, $b = 3$, $c = 4$, then $2c^3 - abc = 2 \times 64 - 2 \times 3 \times 4 = 128 - 24 = 104$.

Examples—22.

Find the numerical values of the following expressions when $a = 1$, $b = 2$, $c = 3$, $d = 4$, $f = 5$:

1. $a - b$.
2. $-a - b$.
3. $-a - b - c$.
4. $a - b - c - d$.
5. $a - b - c + d - f$.
6. $c + d + b - a - f$.
7. abc; $abcd$; $ab + ac - bc \div 2bd$.
8. $5a^2 + b^2 + c^2$; $3a^2 + b^2 + c^2 - 6d^2 - 5f$.

Find the numerical values of the following expressions when $a = 10$, $b = 4$, $c = 3$, $d = 2$, $e = 0$, $f = 1$.

9. $\dfrac{a}{b} + \dfrac{a}{d} - \dfrac{abc}{f}$.

10. $4a + 2b - (3c - 2f)$.

11. $a - (b - c - d)$; $2a + (b - c - d)$.

12. $\dfrac{a + b}{c - a} + \dfrac{c + d}{a - b}$.

13. $\dfrac{a}{b} + \dfrac{a}{c} - \dfrac{a}{bc}$; $bcd - \dfrac{3bd}{4ac}$.

14. $(a + b)^2 - a^2 - b^2$; $(a + b)^2 - (a - b)^2$.

15. $\frac{ef}{a^2} - \frac{c^2}{d^4} - \frac{f^2}{c^3}$.

16. $a(b + c) + b(a - c) - c(f - a)$; $(a - 2)(b - c)$.

17. $5 - \frac{a - b}{c}$; $\frac{3a}{4} + \frac{2b}{3} - \frac{c}{6}$.

18. $4\left(\frac{3}{8(a - 1)} - \frac{1}{8(a + 1)}\right)$.

SECTION XVIII.

SIMPLE EQUATIONS.

52. A principal object of algebra, as of arithmetic, is to find from numbers which are known other numbers which are unknown.

53. To do this, we put a letter for the unknown number, then make an equality from the given conditions; this we call an *equation,* and from it find the value of the unknown letter.

For example: If $x + 4 = 9$, then $x = 9 - 4$, or $x = 5$.

54. **Note.**—All expressions with the sign = between them are not, however, *equations* in the above sense, but are sometimes *identities,* that is, equalities in which both sides are the same.

Thus, $3 + 5 = 8$ is an *identity;* $x + 3x = 4x$ is an identity. So, too, $(x + a)^2 = x^2 + 2ax + a^2$ is an *identity*, and is also called a *formula*. In such expressions x may have any value.

55. If $x + 4 = 9$, x can have but one value, 5.

Definition.—*A simple equation with one unknown letter consists of two expressions with the sign = between them, in which the unknown letter has a determinate value.*

56. To *solve an equation* is to find the *value* of *this unknown letter.*

57. The equation is *satisfied*, or the solution is *verified*, when this value put for the unknown letter in the equation makes it an identity.

Thus, $x + 4 = 9$, gives $x = 5$; and 5 put for x in the equation gives $5 + 4 = 9$, an identity, and the *equation* is *satisfied.*

58. The expression on the *left-hand side* of the sign = is called the *first side.* The expression on the right-hand side is called the *second side* of the equation.

59. Axioms concerning equations.

Axiom 1.—*Both sides of an equation may be multiplied or divided by the same number, and the equality still subsists.*

Axiom 2.—*The same number may be added to or subtracted from both sides of an equation, and the equality still subsists.*

The equality still subsists when we change the signs of all the terms on both sides, as this is simply multiplying both sides by -1.

60. Transposition.—To transpose a term is to change it from one side of an equation to the other.

Rule.—*When we transpose a term, we must at the same time change its sign.*

This is the same as adding the same number to, or subtracting the same number from, both sides.

For example, if we subtract 6 from both sides of the equation

$$x + 6 = 15,$$

we have $$x + 6 - 6 = 15 - 6,$$

or $$x = 15 - 6.$$

Thus, 6 has been transposed from the first side to the second, and its sign changed from $+$ to $-$.

So, also, in $6x = 2 - 3x$. To transpose $3x$ to the first side is the same as adding $3x$ to both sides.

Thus, $$6x + 3x = 2 - 3x + 3x,$$

or $$6x + 3x = 2.$$

So $3x$ is transposed and its sign changed.

SOLUTION OF EQUATIONS.

Equations without Fractions.

61. If the equation has no fractions, it is solved by the following

Rule.—1. *Transpose the unknown letters to the first side, and the numbers or known terms to the second side.*

2. *Apply the rule of addition to the terms collected on the two sides.*

3. *Then divide both sides by the coefficient of* x. (Axiom 1.)

EXAMPLE. Given $12x - 27 = 37 - 3x + 41$, to find x.

Transposing, $12x + 3x = 37 + 27 + 41.$

Adding collective terms, $15x = 105.$

Dividing by 15, $x = 7.$

Examples—23.

Find the value of x in each of the following equations :

1. $x + 4 = 10$; $x + 24 = 20$.
2. $2x + 5 + 8 = x + 14 + 2$; $9x = 72$.
3. $2x + 3x = 55$; $16x - 2x - 6x = 25 + 4x$.
4. $x - 12.5 = 13.6$; $12x = 104$.
5. $1 + 3x + 3 + 5x = 5 + 7x + 7 + 9x$.

Equations with Letters for the Known Numbers.

62. We often have equations in which the known numbers are represented by the letters a, b, c, etc. (first letters of the alphabet), as the unknown are represented by x, y, z. etc., the last letters.

Ex. 1. $4x + 3a - 2b = 12a + x - 8b$.

Transposing, $4x - x = 12a - 3a + 2b - 8b.$

Collecting, $3x = 9a - 6b.$

Dividing by 3, $x = 3a - 2b.$

Ex. 2. $mx = nx + c$.

Transposing, $mx - nx = c.$

Collecting, $(m - n)x = c.$

Dividing by coefficient (co-factor) of x, we have

$$x = \frac{c}{m - n}.$$

Examples—24.

Find x in the equations :

1. $x + a = b$; $x - a = c$.
2. $2x - 2a + c = b$; $ax = c$.
3. $ax - bx + p = m - n$.
4. $3x + b - a = 5x + c$.

Equations with Terms in Brackets.

63. When the equation contains *terms in brackets,* the brackets must be removed, as in Art. 20.

Ex. 1. $6x - (2x - 18) = 22$.

Removing the brackets,

we have	$6x - 2x + 18 = 22$.
Transposing,	$6x - 2x = 22 - 18$.
Reducing,	$4x = 4$,
and	$x = 1$.

Examples—25.

Solve the equations :

1. $5x - (3 + 2x) = 12$; $x - 9 = 5(x - 5)$.
2. $5(x - 6) = - 40$; $30 - 2x = 6x - (24 - x)$.
3. $1 + 3x - (2x - 7) = 10$.
4. $5x + 3(4x - 15) = 50 - 2x$.

Equations with Fractions.

46. If the equation has fractions, we first get rid of the fractions. This is done by the following

Rule.—*Apply the process for reducing all the terms on*

both sides to the least common denominator, dropping the common denominator.

Ex. 1. $\frac{x}{4}+\frac{5x}{6}-\frac{3x}{8}=4.$

The L. C. D. is 24.

Hence we have $\quad 6x+20x-9x=96.$

Collecting, $\quad 17x=96,$

and $\quad x=\frac{96}{17}=5\frac{11}{17}.$

Ex. 2. $\frac{2x}{3}+\frac{x-2}{2}=4-\frac{x-3}{3}.$

Clearing of fractions,

$$4x+3(x-2)=24-2(x-3),$$

or $\quad 4x+3x-6=24-2x+6.$

Collecting, $\quad 9x=36,$

and $\quad x=4.$

Ex. 3. $\frac{x-1}{x-2}=\frac{4x-5}{4x-7}.$

Clearing of fractions, $\quad (x-1)(4x-7)=(4x-5)(x-2).$

Removing brackets (Art. 21), we have

$$4x^2-11x+7=4x^2-13x+10.$$

Striking out the common term $4x^2$ from both sides, we have

$$-11x+7=-13x+10.$$

Transposing and reducing,

$$2x=3,$$

and $\quad x=\frac{3}{2}.$

Examples—26.

Find x in the following equations:

1. $\frac{2x}{3} + \frac{4x}{5} = 22$; $\frac{x}{4} + \frac{x}{5} = 27$.

2. $\frac{3x}{4} + \frac{2x}{5} = 23$.

3. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 156$.

4. $\frac{3x}{2} - \frac{4x}{9} + \frac{5x}{6} - 2 = 32$.

5. $\frac{2x}{3} + \frac{3x}{5} - \frac{4x}{15} = 2\frac{2}{3}$.

6. $a + \frac{x}{3} - \frac{x}{9} = \frac{3a}{2} - \frac{2x}{5} + \frac{a}{5}$.

7. $\frac{x + 12}{7} - \frac{x - 10}{10} = \frac{1}{2}$.

8. $\frac{2x}{3} + \frac{x - 2}{2} = 4 - \frac{x - 3}{3}$.

9. $\frac{3x + 4}{5} - \frac{7x - 3}{2} = \frac{x - 16}{4}$.

10. $\frac{1}{4}(x + 6) - \frac{1}{12}(16 - 3x) = 4\frac{1}{6}$.

11. $\frac{2}{x} - \frac{5}{4x} = \frac{4}{5x} - \frac{1}{20}$

12. $\dfrac{6}{x-2}=\dfrac{5}{x-3}$; $\dfrac{12}{2x+3}=\dfrac{21}{4x-5}$.

13. $\dfrac{2x-3}{3x-4}=\dfrac{2x-4}{3x-5}$.

14. $\dfrac{6x-4}{21}+\dfrac{x-2}{5x-6}=\dfrac{2x}{7}$.

SECTION XIX.

TRANSLATION OF ORDINARY LANGUAGE INTO ALGEBRAIC EXPRESSIONS.

65. As an introduction to the solution of problems by algebra, we give some examples of translations from ordinary language into algebraic expressions.

Ex. 1. A man has x dollars, and gains two dollars. How much has he altogether? *Ans.* $x+2$ dollars.

Ex. 2. A man has x dollars, and loses 20. How much has he left? *Ans.* $x-20$.

Ex. 3. A man has x dollars, what is one-fourth of it? one-third of it? etc. *Ans.* $\dfrac{x}{4}$, $\dfrac{x}{3}$, etc.

Ex. 4. If x is the price of a dozen oranges, what is the price of nine? *Ans.* $\dfrac{9x}{12}$.

Ex. 5. A number exceeds x by 7. What is it? *Ans.* $x+7$.

Ex. 6. x exceeds a number by 7. What is that number?

Ans. $x - 7$.

Ex. 7. A man has x dollars, and loses 3 of them, and then loses $\frac{1}{3}$ of what is left. How much has he after both losses?

Ans. $\frac{2}{3}(x-3)$.

Ex. 8. x dollars at 6 per cent. interest will yield how much in one year?

Ans. $\dfrac{6x}{100}$.

Will amount to how much?

Ans. $x + \dfrac{6x}{100}$.

Ex. 9. A man having x dollars gave away $\frac{1}{2}$ of it, $\frac{1}{3}$ of it, and $\frac{1}{12}$ of it; how much had he left?

Ans. $x - \left(\dfrac{x}{2} + \dfrac{x}{3} + \dfrac{x}{12}\right)$ or $x - \dfrac{11x}{12}$, *i. e.* $\dfrac{x}{12}$.

Ex. 10. 500 is divided into two parts, one of which is x, what is the other?

Ans. $500 - x$,

Ex. 11. If a man goes x miles in 8 hours, how many miles per hour does he travel?

Ans. $\dfrac{x}{8}$.

Ex. 12. If a man goes 15 miles in x hours, how many miles per hour?

Ans. $\dfrac{15}{x}$.

Ex. 13. If a man buy x yards of cloth for \$10, what is the price per yard?

Ans. $\dfrac{10}{x}$.

Ex. 14. If a man pay x dollars a yard for \$15 worth of cloth, what is the number of yards bought?

Ans. $\dfrac{15}{x}$.

Ex. 15. A man goes x miles at the rate of 5 miles an hour; what is the number of hours?

Ans. $\frac{x}{5}$.

Ex. 16. If a man does a piece of work (working uniformly) in x hours, how much of it does he in 1 hour, in 2 hours, in 3 hours, in 6 hours, etc.?

Ans. $\frac{1}{x}$, $\frac{2}{x}$, $\frac{3}{x}$, $\frac{6}{x}$, etc.

Ex. 17. If a man does a piece of work in 10 days, how much of it does he in x days?

Ans. $\frac{x}{10}$.

Ex. 18. A pipe fills a cistern in 8 hours, what fraction of it does it fill in x hours?

Ans. $\frac{x}{8}$.

Ex. 19. What number bears to x the proportion of 3 to 4?

Ans. $\frac{3x}{4}$.

Ex. 20. If two numbers bear to each other the ratio of 4 to 5, and one of them is $4x$, what is the other?

Ans. $5x$.

NOTE.—Consecutive numbers are numbers each of which is greater by unity than the preceding one; thus, 2, 3, 4 are consecutive numbers.

Ex. 21. Write four consecutive numbers of which x is the smallest. *Ans.* x, $x + 1$, $x + 2$, $x + 3$.

Ex. 22. Write three consecutive numbers of which x is the greatest. *Ans.* $x - 2$, $x - 1$, x.

Ex. 23. If x is the tens figure, and y the units figure of a number, what is the number? *Ans.* $10x + y$.

Ex. 24. If x be the number of minute spaces moved over by the minute hand of a watch in a certain time, what number of spaces does the hour hand move over at the same time?

Ans. $\dfrac{x}{12}$.

Ex. 25. A man travels 25 miles in 6 hours, how many miles does he travel in x hours at the same rate?

Ans. $\dfrac{25x}{6}$.

SECTION XX.

PROBLEMS IN SIMPLE EQUATIONS.

66. To form an equation we follow this

Rule.—*Represent the unknown quantity by* x; *then form the expressions and the equation according to the conditions of the problem.*

The equation thus formed is solved as explained in Section XVIII.

Ex. 1. What number is that to which if 5 be added $\frac{1}{3}$ of the sum will be 25?

Let $x =$ the number.

Adding 5 to it, we have $x + 5$, and $\frac{1}{3}$ of this is $\frac{1}{3}(x + 5)$.

By the condition, $\frac{1}{3}(x + 5) = 25$.

Hence, $x + 5 = 75$,

and $x = 70$.

Verification. $\frac{1}{3}(70 + 5) = 25$,

or $25 = 25$, an identity.

Ex. 2. What number is it of which the third and fourth parts together make 21 ?

Let $x =$ the number.

Then $\frac{x}{3} =$ its third part,

and $\frac{x}{4} =$ its fourth part.

Then by the question $\frac{x}{3} + \frac{x}{4} = 21.$

Reducing, $4x + 3x = 21 \times 12,$

or $7x = 252,$

and $x = 36.$

Ex. 3. Find two consecutive numbers such that $\frac{1}{4}$ of the smaller added to $\frac{1}{3}$ of the greater is equal to 5.

Let $x =$ the smaller ;

then $x + 1 =$ the greater.

Hence we have $\frac{x}{4} + \frac{x+1}{3} = 5.$

Reducing, $7x = 56,$

and $x = 8.$

Hence, $x + 1 = 9$, and the two numbers are 8 and 9.

Ex. 4. Divide 54 into two parts, one of which shall be to the other as 4 : 5.

Let $4x =$ one part,

and $5x =$ the other part.

Then $4x + 5x = 54.$

$\therefore x = 6$; $4x = 24$, one part ; and $5x = 30$, the other part.

Verification. $30 + 24 = 54.$

$$\frac{24}{30} = \frac{4}{5}.$$

Ex. 5. Divide eight dollars and a half into the same number of dimes, half-dollars, and quarters.

Let $x =$ the number of each.

Then $10x =$ the value of the dimes in cents,

$50x =$ " of the half-dollars in cents,

$25x =$ " of the quarters in cents.

Therefore, $10x + 50x + 25x = 850$;

whence $x = 10.$

Hence, 10 dimes, 10 half-dollars, and 10 quarters make eight dollars and a half.

Examples—27.

1. What number is that which multiplied by 7 is greater by 12 than 51?

2. What number is it, $\frac{9}{10}$ of which is 3 greater than 15?

3. A train has 15 more freight cars than passenger cars, and 33 cars in all. How many of each sort in it?

4. A garrison of 3,280 men has 3 times as many artillerists as cavalry men, and 4 times as many infantry as artillerists. How many of each of these men?

5. What number is it $\frac{1}{3}$ of which added to $\frac{1}{12}$ of it is equal to 20?

6. The difference between $\frac{1}{5}$ and $\frac{1}{6}$ of a number added to $\frac{1}{3}$ of it is 22, what is the number ?

7. Find two consecutive numbers of which $\frac{1}{7}$ of the greater subtracted from $\frac{1}{6}$ of the lesser leaves 1.

8. Divide 45 into three parts which shall be consecutive numbers.

9. Find two consecutive numbers of which the lesser diminished by 8 is one-half the greater.

10. The sum of two numbers is 24, and 9 times the one is equal to 3 times the other. Find the numbers.

11. The difference of two numbers is 12, and 4 added to twice the smaller gives the greater. Find the two numbers.

12. I paid $34 in half-dollars, quarters, and dimes, and used the same number of each of these coins. What was this number ?

13. A father is now 4 times as old as his son. Five years ago he was 5 times as old; what is now the age of each ?

14. A company consists of 90 persons ; the men are 4 more than the women, and the children 10 more than the grown persons. Find the number of each.

15. The $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$ of a certain sum of money are together $4 more than the amount itself. What is it ?

SECTION XXI.

PROBLEMS—Continued.

67. Ex. 1. A can do a piece of work in 4 days, and B can do it in 3 days. In what number of days will they both together do it?

Let the work be 1, and x = the required number of days.

In x days A does $\frac{x}{4}$ of the work.

In x days B does $\frac{x}{3}$ of the work.

Hence, $$\frac{x}{4} + \frac{x}{3} = 1,$$

or $$7x = 12,$$

and $$x = 1\tfrac{5}{7} \text{ days}.$$

Ex. 2. A, B, and C divide 700 acres; A taking 4 acres to B's 5, and 3 acres to C's 2. How many acres did each get?

Let x = A's number of acres;

then $\frac{5}{4}x$ = B's " "

and $\frac{2}{3}x$ = C's " "

Hence, $$x + \tfrac{5}{4}x + \tfrac{2}{3}x = 700.$$

Solving, $$x = 240 = \text{A's acres.}$$

$$\tfrac{5}{4}x = 300 = \text{B's} \text{ "}$$

$$\tfrac{2}{3}x = 160 = \text{C's} \text{ "}$$

Ex. 3. A man had \$2,000, a part of which he lent at 4 per cent. per annum, and a part at 6 per cent. The annual income from the whole was \$92. Find the two parts.

Let $x =$ the number of dollars lent at 4 per cent. This produces $\frac{4x}{100}$ dollars per annum.

$2000 - x =$ the number of dollars lent at 6 per cent. and this yields $\frac{6}{100}(2000 - x)$ dollars per annum.

Therefore, $$\frac{4x}{100} + \frac{6}{100}(2000 - x) = 92;$$

whence $$x = 1400 \text{ dollars},$$

and $$2000 - x = 600 \text{ dollars}.$$

Examples—28.

1. Divide 102 into 4 parts which shall be consecutive numbers.

2. A cistern is filled by one pipe in 8 hours, and by another in 3 hours—in what time will it be filled if both pipes run at the same time?

3. Find a number such that if the half of it be taken from 36, $\frac{3}{10}$ of the remainder will be equal to $\frac{4}{5}$ of the original number.

4. A and B being on the same road 21 miles apart, they set out at the same hour towards each other, A walking 3 miles an hour and B at the rate of 4 miles an hour. How many hours will elapse ere they meet, and how far will each have walked?

5. A can do a piece of work in 5 days, B in 6 days, and C in 8 days. In what time can they do it all working together?

6. A farmer had two flocks of sheep of the same number. He sold 39 from one flock, and 93 from the other, and found he had remaining in one flock twice as many as in the other. How many were in the flocks at the beginning?

7. A sets out for a town 12 miles off, and walks at the rate of 4 miles an hour. Half an hour afterward B sets out from the same place, in the same direction, running 5 miles an hour. How far from the town will B overtake A?

8. The denominator of a certain fraction exceeds the numerator by 2, and if 2 be subtracted from the numerator and added to the denominator, the new fraction thus formed is equal to $\frac{1}{3}$. What is the fraction?

9. Seven maidens met a boy who was carrying a basket of apples. One maiden bought $\frac{2}{7}$ of the apples; the second, $\frac{1}{12}$; the third, $\frac{1}{6}$; and the fourth, $\frac{1}{3}$ of them; the fifth bought 20 apples; the sixth bought 12, and the seventh bought 11, and this left the boy one apple. How many had he at first, and how many did the first four maidens take?

10. Polycrates, the tyrant of Samos, asked Pythagoras the number of his pupils. Pythagoras answered him: The half of them study mathematics; one-fourth part study the secrets of nature; the seventh part listen to me in silent meditation, and then there are three more, of whom Theano excels them all. This will give you the number of pupils whom I am guiding to the boundaries of immortal truth.

SECTION XXII.

SIMPLE EQUATIONS WITH TWO UNKNOWN QUANTITIES.—ELIMINATION.

68. Simultaneous Equations are those which are true for the *same* values of the unknown quantities which they contain.

If $\quad x - y = 5, \quad$ or $\quad x = y + 5 \quad . \quad . \quad . \quad . \quad (1).$

Then, when $y = 1, \qquad x = 6;$

$y = 2, \qquad x = 7;$

$y = 3, \qquad x = 8,$ etc., indefinitely.

Now suppose, at the same time,

$$x + y = 9 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Then among the values of x and y in the above list, only $x = 2$, $y = 7$ will satisfy equation (2), and, therefore, $\left.\begin{matrix} x = 2 \\ y = 7 \end{matrix}\right\}$ are the only values which belong to both of the equations $\left.\begin{matrix} x - y = 5 \\ x + y = 9 \end{matrix}\right\}$ taken together, or simultaneously.

Hence, *Two simultaneous simple equations with two unknowns give one value for each of the two unknowns.*

69. Elimination.—To find these values, we first reduce the two equations to a single equation containing only one of the unknown letters. This process is called elimination. That is, we first *eliminate,* or get rid of one of the unknown letters. Three methods of *elimination* are usually given.

70. First Method.—*Multiply the given equations by*

such numbers as will make the coefficients of one of the unknowns the same in both. Then add or subtract the equation thus obtained, according as these equal coefficients have contrary or the same signs. We thus get a simple equation with one unknown letter.

Ex. 1. Given $\left.\begin{aligned} 2x + 3y &= 9 \quad (1) \\ 4x - 5y &= 7 \quad (2) \end{aligned}\right\}$ to find x and y.

To *eliminate* y, multiply equation (1) by 5, and equation (2) by 3. We have this.

$$\begin{aligned} 10x + 15y &= 45 \\ 12x - 15y &= 21 \end{aligned}$$

Adding, we get $$22x = 66$$

$$\therefore \quad x = 3.$$

We may find y by *eliminating* x, or more simply thus :

Since $x = 3$, $2x = 6$; and hence, substituting 6 for $2x$ in (1), we get

$$6 + 3y = 9,$$

or $$y = 1.$$

Therefore, $\left.\begin{aligned} x &= 3 \\ y &= 1 \end{aligned}\right\}$ are the required values of x and y.

Ex. 2. Given $\left.\begin{aligned} 2x + 3y &= 13 \quad (1) \\ 4x + 2y &= 14 \quad (2) \end{aligned}\right\}$ to find x and y.

To eliminate x, we have

$$\begin{aligned} 4x + 6y &= 26 \\ 4x + 2y &= 14 \end{aligned}$$

Subtracting, we get $$4y = 12$$

or $$y = 3.$$

And putting 3 for y in (1), we have

$$2x + 9 = 13,$$

or $$x = 2.$$

71. Second Method.—*Find the expression for one of the unknown letters, as* x, *in terms of the other, in one of the equations, and substitute this for* x *in the other equation.*

Ex. 3. $3x - y = 6 \quad . \ . \ . \ . \ . \ . \ . \ . \quad (1).$

$5x + 2y = 32 \quad . \ . \ . \ . \ . \ . \ . \quad (2).$

The expression for x in (1) is $x = \frac{y + 6}{3}$.

Putting this for x in (2), we have

$$5\left(\frac{y+6}{3}\right) + 2y = 32,$$

or

$$5y + 30 + 6y = 96,$$

$$11y = 66,$$

and

$$y = 6.$$

Again, putting 6 for y in (1), we have $3x - 6 = 6$,

or $\quad 3x = 12 \quad$ and $\quad x = 4.$

72. Third Method.—*Express one of the unknown letters in terms of the other in each equation, and put these expressions equal to one another.*

Ex. 4. Given $4x + 3y = 19 \quad . \ . \ . \ . \ . \quad (1).$

$2x - 5y = 3 \quad . \ . \ . \ . \ . \quad (2).$

From (1), $\quad 3y = 19 - 4x, \quad$ or $\quad y = \frac{19 - 4x}{3} \quad . \ . \ . \ . \ . \quad (3).$

From (2), $\quad 5y = 2x - 3, \quad$ or $\quad y = \frac{2x - 3}{5} \quad . \ . \ . \ . \ . \quad (4).$

Equating (3) and (4), $\quad \frac{19 - 4x}{3} = \frac{2x - 3}{5}$,

or

$$95 - 20x = 6x - 9.$$

Hence,

$$26x = 104,$$

$$x = 4.$$

Putting 4 for x in (2), we get $8 - 5y = 3$; hence, $y = 1$.

Examples—29.

Solve the simultaneous equations

$$\left.\begin{aligned} x + y &= 15 \\ 6x &= 4y \end{aligned}\right\} (1). \qquad \left.\begin{aligned} x + y &= 24 \\ x - y &= 10 \end{aligned}\right\} (2).$$

$$\left.\begin{aligned} x - y &= 30 \\ 2x + y &= 120 \end{aligned}\right\} (3). \qquad \left.\begin{aligned} x + 5y &= 14 \\ 3x - 4y &= 4 \end{aligned}\right\} (4).$$

$$\left.\begin{aligned} 5x + 9y &= 65 \\ 7x + 3y &= 43 \end{aligned}\right\} (5). \qquad \left.\begin{aligned} 8x + 3y &= 113 \\ 12x - 9y &= 219 \end{aligned}\right\} (6).$$

$$\left.\begin{aligned} 7x + 2y &= 85 \\ 18x - 3y &= 129 \end{aligned}\right\} (7). \qquad \left.\begin{aligned} \frac{x}{4} + \frac{y}{3} &= 6 \\ \frac{x}{2} + \frac{y}{4} &= 7 \end{aligned}\right\} (8).$$

$$\left.\begin{aligned} \frac{x}{11} + 2y &= 15 \\ 5x + \frac{3y}{7} &= 58 \end{aligned}\right\} (9). \qquad \left.\begin{aligned} 16x + 17y &= 274 \\ 24x - 105y &= 150 \end{aligned}\right\} (10).$$

$$\left.\begin{aligned} 7x - 3y &= 0 \\ 2x + 5y &= 41 \end{aligned}\right\} (11). \qquad \left.\begin{aligned} 5(x+2) - 3(y+1) &= 23 \\ 3(x-2) + 5(y-1) &= 19 \end{aligned}\right\} (12).$$

$$\left.\begin{aligned} \frac{x}{3} + \frac{7y}{10} &= 670 \\ \frac{3x}{4} - y &= 1250 \end{aligned}\right\} (13). \qquad \left.\begin{aligned} 3x &= 2y + 14 \\ \frac{75}{x-y} &= \frac{200}{x} \end{aligned}\right\} (14).$$

$$\left.\begin{aligned}\frac{x+2}{3}+\quad 8y&=\ 31\\ \frac{y+5}{4}+10x&=192\end{aligned}\right\}(15).\qquad \left.\begin{aligned}1.7x-2.2y&=-7.9\\ 4y-\quad 5x&=1\end{aligned}\right\}(16).$$

$$\left.\begin{aligned}\frac{x-1}{3}-\frac{y+2}{4}&=\frac{2(x-y)}{5}\\ \frac{x-3}{4}-\frac{y-3}{4}&=2y-x\end{aligned}\right\}(17).\qquad \left.\begin{aligned}\frac{x}{3}-y&=\ 0\\ x-\frac{y}{2}&=10\end{aligned}\right\}(18).$$

SECTION XXIII.

PROBLEMS PRODUCING SIMULTANEOUS EQUATIONS.

73. Ex. 1. The sum of two numbers is 21, and their difference is 5. Find the numbers.

Let $\qquad x =$ one number,

and $\qquad y =$ the other number.

Then
$$\left.\begin{aligned}x+y&=21\\ x-y&=5\end{aligned}\right\}.$$

Adding, $\qquad 2x = 26,$

and $\qquad x = 13.$

Putting 13 for x, $\qquad 13 + y = 21.$

From which we find $\qquad y = 8.$

Ex. 2. The sum of two numbers is 44, and if $\frac{1}{3}$ of the less be added to $\frac{1}{4}$ of the greater, the result is 12. Find the numbers.

Let $x =$ the greater number,

and $y =$ the less number.

Then

$$\left.\begin{aligned} x + y &= 44 \\ \tfrac{1}{4}x + \tfrac{1}{3}y &= 12 \end{aligned}\right\}.$$

From which we find $x = 32$,

$y = 12$.

Ex. 3. A certain number consists of two digits. The sum of the digits is 7, and if 45 be added to the number, we get a number the digits of which are the digits of the first number reversed. Find the number.

Let $x =$ the tens digit,

and $y =$ the units digit.

Then $10x + y =$ the number,

and we have

$$\left.\begin{aligned} x + y &= 7 \\ 10x + y + 45 &= 10y + x \end{aligned}\right\},$$

which give $x = 1, \quad y = 6$;

or the number is 16; and $16 + 45 = 61$, the digits of 16 reversed.

Examples—30.

1. Find two numbers such that $\frac{1}{3}$ of the one added to $\frac{1}{4}$ of the other shall be 20, but $\frac{1}{3}$ of the latter added to $\frac{1}{4}$ of the former shall be equal to 22.

2. On adding 18 to a certain number of two digits, we get a number in which these digits are reversed, and the two numbers added together make 44. What is the first number?

3. Divide 46 into two such parts that when the greater part is divided by 7, and the smaller part by 3, the quotients together make 10.

4. A said to B, give me $\frac{3}{4}$ of your money and I will have \$100. B said to A, give me $\frac{1}{2}$ of yours and I will have \$100. Find how much A and B had at first.

5. The difference between two sums put out at interest for one year is \$2,000. One sum is put out at 5 per cent. and the other at 4 per cent., and the incomes from them are equal. Find the two sums.

6. A merchant has his house and goods together insured for \$36,000. His house at $1\frac{1}{2}$ per cent. premium, and his goods at 2 per cent. The difference of the two premiums amounts to \$6.75. What was the amount of insurance on the house, and what on the goods?

7. What fraction is that which is equal to $\frac{1}{10}$ when 2 is subtracted from its numerator, and equal to $\frac{1}{6}$ when 4 is added to its denominator?

8. If the number of cows in a field were doubled, then there would be 84 cows and horses together in the field. If the number of horses be doubled, and $\frac{2}{3}$ of the cows be taken out there would be 80 horses and cows in the field. What is the number of horses and cows in the field?

9. The sum of two numbers is 30, and their quotient is 4. Find the numbers.

10. In the second class of a school there are $1\frac{11}{16}$ times as many pupils as there are in the first class. From the first class 7 go away, and 6 enter the second class, and then the second has $2\frac{2}{5}$ as many as the first. How many in each class?

SECTION XXIV.

SIMULTANEOUS SIMPLE EQUATIONS OF THREE OR MORE UNKNOWN QUANTITIES.

74. If three equations are given, with three unknown quantities, we may eliminate one of the unknown quantities, and thus obtain two equations with two unknowns, and then from these two we may obtain one equation with one unknown, by the methods of Arts. 70, 71, 72.

EXAMPLE. Given

$$\left.\begin{aligned} 7x + 2y + 3z &= 20 \quad (1) \\ 3x - 4y + 2z &= 1 \quad (2) \\ -2x + 5y + 7z &= 29 \quad (3) \end{aligned}\right\}, \text{ to find } x,\ y, \text{ and } z.$$

First, to eliminate y between (1) and (2),

$$\begin{array}{lrl} \text{we have} & 14x + 4y + 6z &= 40 \\ & 3x - 4y + 2z &= 1 \\ \hline \text{Hence,} & 17x \quad + \quad 8z &= 41 \quad (4) \end{array}$$

Second, to eliminate y between (1) and (3),

we have $$35x + 10y + 15z = 100$$
$$-4x + 10y + 14z = 58$$

Hence, $$39x + z = 42 \quad (5).$$

Third, to eliminate z between (4) and (5),

we have $$312x + 8z = 336$$
$$17x + 8z = 41$$

Hence, $$295x = 295,$$

or $$x = 1.$$

Then, from (4) $$17 + 8z = 41,$$

or $$8z = 24,$$

and $$z = 3.$$

And from (2), $$3 - 4y + 6 = 1,$$

or $$4y = 8,$$

and $$y = 2.$$

Examples—31.

Solve the simultaneous equations

$$\left.\begin{aligned} 5x + 4y - 2z &= 14 \\ 3x + 2y + z &= 16 \\ x - 9y + 8z &= 7 \end{aligned}\right\} (1).$$

$$\left.\begin{aligned} x + y + z &= 0 \\ x - y + z &= 4 \\ 5x + y + z &= 20 \end{aligned}\right\} (2).$$

$$\left.\begin{aligned} x + y &= 8 \\ y + z &= 6 \\ x + z &= 10 \end{aligned}\right\} (3).$$

$$\left.\begin{aligned} y - x &= \tfrac{1}{3}z \\ z - x &= \tfrac{1}{2}y \\ x + z &= 2(y - 1) \end{aligned}\right\} (4).$$

$$\left.\begin{aligned} \frac{x}{2} + \frac{y}{3} &= 9 \\ \frac{y}{3} + \frac{z}{5} &= 8 \\ \frac{z}{5} + \frac{x}{2} &= 5 \end{aligned}\right\} (5).$$

$$\left.\begin{aligned} 3x + 4y + z &= 14 \\ 2x + y + 5z &= 19 \\ 5x + 2y + 3z &= 18 \end{aligned}\right\} (6).$$

$$\left.\begin{aligned} 40x - 16y + 25z &= 6 \\ 12x + 4y + 20z &= 62 \\ 36x + 17y - 15z &= 56 \end{aligned}\right\} (7).$$

SECTION XXV.

INVOLUTION OR RAISING TO POWERS.

75. Involution is the process of raising quantities to powers. It is multiplication, therefore, in which the factors are all equal, and requires no rules different from those already given (Section IV.).

76. We will notice some results, however, which will be of use in shortening the process.

(1.) **Rule of the Signs.**—*Any even power of a minus quantity is plus. And any odd power of a minus quantity is minus.*

Thus, the square of $-a$ is $-a \times -a = +a^2$.

The fourth power of $-a$ is $-a \times -a \times -a \times -a = +a^4$.

The third power of $-a$ is $-a \times -a \times -a = -a^3$.

The fifth power of $-a$ is $-a \times -a \times -a \times -a \times -a = -a^5$.

(2.) **Rule of the Exponents.**—*To raise a quantity to a power, multiply its exponent by the exponent of the required power.*

Thus, $(a^2)^2 = a^2 \times a^2 = a^4 = a^{2\times 2}$;

$(a^3)^2 = a^3 \times a^3 = a^6 = a^{3\times 2}$;

$(a^4)^2 = a^4 \times a^4 = a^8 = a^{4\times 2}$.

77. Powers of Monomials.—To raise a monomial to a power,

Rule.—*Raise the coefficient to the required power, observing the rule of the signs, and multiply the exponent of each letter by the exponent of the power.*

Thus, the square of a^2b^3 is a^4b^6.

The third power of $-2ab$ is $-8a^3b^3$.

The fourth power of $2ab^2c^3$ is $16a^4b^8c^{12}$.

78. Squares of Monomials.—As we will have mainly to do with squares in this book, we will particularize for this case. Therefore, to square a monomial,

Rule.—*Square the coefficient and multiply the exponents by two.*

Thus, the square of $5a^3b^4c$ is $25a^6b^8c^2$.

79. Powers of Fractions.—To raise a fraction to a power

Rule.—*Raise the numerator and denominator to the required power, observing the rule of the signs.*

Thus, $\left(\frac{2}{3}\right)^2 = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$.

$$\left(\frac{a^2}{b^3}\right)^2 = \frac{a^2}{b^3} \times \frac{a^2}{b^3} = \frac{a^4}{b^6}.$$

$$\left(-\frac{2a}{3b}\right)^3 = -\frac{8a^3}{27b^3}.$$

80. Squares of Fractions.—As a particular case, to square a fraction,

Rule.—*Square the numerator and denominator.*

81. Powers of Polynomials.—To raise a polynomial to any power, we simply perform the multiplications indicated by the exponent of the required power (always one less than the exponent).

Thus, $(a + b)^2 = (a + b)(a + b)$.

$$(a + b)^3 = (a + b)(a + b)(a + b).$$

$$(a + b)^4 = (a + b)^2 \times (a + b)^2 =$$

$$(a + b)(a + b)(a + b)(a + b).$$

82. Squares of Polynomials.—For the squares of polynomials, we repeat the two important results of Articles 25, 26.

$$(a + b)^2 = a^2 + 2ab + b^2 = (-a - b)^2.$$

$$(a - b)^2 = a^2 - 2ab + b^2 = (b - a)^2.$$

Also, by performing the multiplications required, we find the following useful results,

$$(a + b + c)^2 = a^2 + 2ab + 2ac + b^2 + 2bc + c^2.$$

$$(a - b + c)^2 = a^2 - 2ab + 2ac + b^2 - 2bc + c^2.$$

Examples—32.

1. Find the third power of $3ab^2$.

2. Find the third power of $-2a^2bc$.

3. Find the fourth power of $\frac{ab^2}{2c^3}$.

Square each of the following quantities:

4. $-8ab$. 5. $8a^2bc^3$. 6. $\frac{3cx}{2ay}$. 7. $-\frac{3y^2}{2x^3}$.

Write the squares of

8. $a+2$. 9. $2bc+1$. 10. $3m-5n$.

11. $acx+y$. 12. $\frac{ab}{3}+c$.

13. Find the square of $2a+3b-c$.

14. Find the third power of $a-b$.

15. Find the fourth power of $a+b$.

SECTION XXVI.

EVOLUTION OR EXTRACTION OF ROOTS.—SQUARE ROOT.

83. Roots of Quantities.—The root of a quantity is the quantity the involution of which produces the given quantity.

Thus, the square root of a^2 is a, because a squared gives a^2; also, the *cube* root of $8a^3$ is $2a$, because $(2a)^3 = 2a \times 2a \times 2a = 8a^3$.

The *fourth* root of $16a^4$ is $2a$, because $(2a)^4 = 2a \times 2a \times 2a \times 2a = 16a^4$.

84. Evolution is the process of finding the roots of quantities, and is the reverse operation of *involution;* and the rule of *evolution,* or *extraction of roots,* is found from the results of raising to powers.

85. The signs of evolution, or of roots to be extracted, are placed on the left of the quantities, and are as follows:

$\sqrt{}$ for "the square root of,"

$\sqrt[3]{}$ for "the cube root of,"

$\sqrt[4]{}$ for "fourth root of," etc., etc.

3, 4, etc., being called *indices* of the roots (the index 2 being understood when none is written).

86. We will notice some results which follow directly from the rules of involution.

(1.) *Any even root of a + quantity may be either + or −, and must, therefore, be written with the double sign* ± (plus or minus).

Thus, $\sqrt{9} = \pm 3, \qquad \sqrt{4a^2} = \pm 2a, \qquad \sqrt[4]{a^4} = \pm a.$

(2.) *Any odd root of a quantity has the same sign as the quantity itself.*

Thus, $$\sqrt[3]{a^3} = +a,$$

and $$\sqrt[3]{-a^3} = -a.$$

(3.) *There can be no even root of a minus quantity.*

Thus, the square root of $-a^2$ cannot be extracted, for

$(+a)^2 = +a^2$, and $(-a)^2 = +a^2$; and, therefore, such expressions as $\sqrt{-a^2}$, are called *impossible*, or *imaginary* quantities.

SQUARE ROOT.

87. Under evolution we shall only give the rules for finding the *Square Root* of algebraic quantities.

88. Square Root of Monomials.—The square of $2a^3b^2$ is $2a^3b^2 \times 2a^3b^2 = 4a^6b^4$; hence the square root of $4a^6b^4$ is $\pm 2a^3b^2$. (Art. 86 (1).)

Therefore, to find the square root of a monomial, we have the

Rule.—*Take the square root of the coefficient, and divide the exponents of the letters by 2. Write the sign ± before the result.*

Note.—A minus quantity has no square root. (Art. 86 (3).)

89. Square Root of Fractions.—The square of $\pm\frac{2a}{3b}$ is $\frac{4a^2}{9b^2}$; hence the square root of $\frac{4a^2}{9b^2}$ is $\pm\frac{2a}{3b}$, or, to extract the square root of a fraction,

Rule.—*Take the square root of the numerator and denominator.*

Ex. 1. The square root of $\frac{16a^6x^2}{81b^2y^2}$ is $\pm\frac{4a^3x}{9by}$.

Ex. 2. Find $\sqrt{\frac{4}{9} + \frac{20}{3}}$.

First add these fractions: $\frac{4}{9} + \frac{20}{3} = \frac{4}{9} + \frac{60}{9} = \frac{64}{9}$.

Hence, $\sqrt{\frac{4}{9} + \frac{20}{3}} = \sqrt{\frac{64}{9}} = \pm\frac{8}{3}$.

Examples—33.

Find the square root of the following quantities:

1. $25a^2y^2$.
2. $100a^4x^2y^2$.
3. $49a^2b^4$.
4. $\dfrac{9a^2x^2}{4b^2y^2}$.
5. $\dfrac{16a^4b^2}{49x^6y^6}$.
6. $\dfrac{9x^2y^4}{4a^2}$.
7. $\dfrac{19a^2}{16} + \dfrac{3a^2}{8}$.
8. $\dfrac{6}{25} + \dfrac{2}{5}$.
9. $\dfrac{4}{9} - \dfrac{1}{3}$.
10. $\dfrac{37}{16} - \dfrac{3}{4}$.

90. Square Root of Trinomials.—Since $(a + b)^2 = a^2 + 2ab + b^2$, the square root of $a^2 + 2ab + b^2$ is $\pm(a+b)$.

Since $(a - b)^2 = a^2 - 2ab + b^2$, the square root of $a^2 - 2ab + b^2$ is $\pm (a - b)$.

Therefore, to find the square root of a trinomial, we have the

Rule.—*Arrange it with reference to one of its letters, then take the square roots of the first and last terms, and put the sign of the middle term between them.*

Ex. 1. The square root of $x^2 + 6x + 9 = \sqrt{x^2} + \sqrt{9} = \pm (x + 3)$.

Ex. 2. The square root of $4a^2 - 12ab + 9b^2 = \sqrt{4a^2} - \sqrt{9b^2} = \pm (2a - 3b)$.

91. Note 1.—Any number or algebraic expression is called a *perfect* or *complete* square when its square root can be exactly found.

Note 2.—A trinomial is a complete square *if, when arranged by one of its letters, the middle term is twice the product of the square roots of the first and last terms; that is, when the square of the middle term is four times the product of the first and last terms.*

Note 3.—Since a *monomial* squared gives a *monomial*, and a *binomial* squared gives a *trinomial*, a binomial cannot be a complete square.

Examples—34.

Find the square root of the following trinomials :

1. $a^2 + 2a + 1$.
2. $x^2 + 4 - 4x$.
3. $x^2 + 5x + \frac{25}{4}$.
4. $9a^2b^2 - 6abx + x^2$.
5. $16a^4 - 24a^2b^2 + 9b^4$.
6. $64x^4 + \frac{9}{4}b^2 - 24bx^2$.
7. $a^2 + \frac{1}{4} + a$.
8. $16x^2 - 8x + 1$.
9. $x^2 - 3x + \frac{9}{4}$.
10. Are $x^2 - 6x + 4$, $a^2b^2 - 2abx + x^4$, $a^2 - \frac{3}{2}a + \frac{9}{4}$ complete squares ?

92. Completing the Square.—In equations we have often to make such algebraic expressions as $x^2 + px$, and

$x^2 - px$, complete trinomial squares by adding the *right term.* This process is called *Completing the Square.*

Examining the trinomial squares given in Art. 90, viz., $a^2 + 2ab + b^2$, $a^2 - 2ab + b^2$, we see that in order to make any binomial expression of the form $x^2 + px$, or $x^2 - px$ a complete trinomial square, we have the

Rule.—*Add the square of half the coefficient or factor of* x *in the second term.*

Ex. 1. To $x^2 + 4x$ add 2^2, and we have $x^2 + 4x + 4$, the square root of which is $\pm (x + 2)$.

Ex. 2. To $x^2 - 3ax$ add $\left(\frac{3a}{2}\right)^2$, and we have $x^2 - 3ax + \frac{9a^2}{4}$, the square root of which is $\pm \left(x - \frac{3a}{2}\right)$.

Ex. 3. To $x^2 - \frac{4}{3}x$ add $(\frac{2}{3})^2$, and we have $x^2 - \frac{4}{3}x + \frac{4}{9}$, the square root of which is $\pm (x - \frac{2}{3})$.

Examples—35.

Complete the squares of the following expressions, and find the square root of each result.

1. $x^2 + 6x$.	2. $x^2 - 12x$.
3. $x^2 - 11x$.	4. $x^2 - x$.
5. $y^2 + 3y$.	6. $x^2 - px$.
7. $a^2 - \frac{8}{3}a$.	8. $x^2 - 4ax$.
9. $y^2 + \frac{4}{7}y$.	10. $x^2 - \frac{7}{6}x$.
11. $x^2 - \frac{9}{10}x$.	12. $x^2 + \frac{5}{8}x$.

93. Square Root of Polynomials.—The rule for the square root of polynomials is similar to the rule in arithmetic.

We know that the square root of $a^2 + 2ab + b^2$ is $a + b$.

Hence, writing down the terms, and proceeding as in arithmetic, we have the following arrangement:

$$\begin{array}{r|l} a^2 + 2ab + b^2 & a + b \\ \cline{2-2} a^2 \qquad\qquad & 2a \\ \cline{1-1} 2a + b\ )\ 2ab + b^2 & \\ 2ab + b^2 & \\ \cline{1-1} \end{array}$$

That is, 1. *Arrange the polynomial with reference to one of its letters.*

2. *Find the square root of the first term, and subtract its square from the polynomial.*

3. *For a divisor double the first term of the root. Divide the first term of the remainder by this divisor; the quotient will be the second term of the root.*

4. *Place this second term in the root, and to the right of the divisor. Multiply the divisor, thus increased, by the second term of the root, and subtract the product from the dividend.*

94. If the root contain more than two terms, a like process continued, *doubling each time the root already found,* will find all the terms.

Thus, we know $(a + b + c)^2 = a^2 + 2ab + b^2 + 2ac + 2bc + c^2$.

Hence, to find the square root of this latter expression, we proceed as in the following arrangement:

$$
\begin{array}{l|l|l}
 & a^2 + 2ab + b^2 + 2ac + 2bc + c^2 & a + b + c \\
 & a^2 & 2a \\
\hline
2a + b & 2ab + b^2 & \\
 & 2ab + b^2 & \\
\hline
2a + 2b + c & 2ac + 2bc + c^2 & \\
 & 2ac + 2bc + c^2 & \\
\hline
\end{array}
$$

Examples—36.

Find the square root of

1. $64a^2 + 144ab + 81b^2$.

2. $x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4$.

3. $x^4 - 4x^3 + 8x + 4$.

4. $9a^2 - 12ab + 4b^2 + 6ac - 4bc + c^2$.

5. $9a^4 - 12a^3 + 10a^2 - 4a + 1$.

6. $16a^4 + 4b^2 + c^4 - 16a^2b + 8a^2c^2 - 4bc^2$.

7. $9c^4 - 6ac^2 + a^2 + 12c^2 - 4a + 4$.

8. $a^2 + b^2 - 2ab + 4ac - 4bc + 4c^2$.

9. $x^4 - 2x^3 - x^2 + 2x + 1$.

10. $\dfrac{x^4}{9} - \frac{1}{3}x^2 + \frac{1}{4}$.

11. $a^4 - 10a^3 + 37a^2 - 60a + 36$.

SECTION XXVII.

QUADRATIC EQUATIONS.

95. A **Quadratic Equation** is an equation of the *second degree*, that is, it contains the *second power of the unknown letter*.

96. There are two sorts of quadratic equations:

1st. *Pure Quadratics*, which contain x^2 and not x.

2d. *Affected Quadratics*, which contain both x^2 and x.

Thus, $5x^2 - 2 = 6x^2 - 5$, $x^2 = 9$, $\dfrac{x^2}{2} + \dfrac{3x^2}{4} = 1$ are pure quadratics;

While $3x^2 + 5x = 6$, $\dfrac{x^2}{2} - 9x = 5$, $ax^2 + bx = c$ are affected quadratics.

97. **Solution of Pure Quadratics.**—To solve a Pure Quadratic,

Rule.—*Find* x^2, *as in simple equations. Then take the square root of both sides, putting the sign* $\pm$ *before the root of the second side.*

Ex. 1. Given $x^2 - \dfrac{3x^2}{4} = 1$.

Clearing of fractions,

$$4x^2 - 3x^2 = 4,$$
$$x^2 = 4,$$
$$x = \pm 2.$$

Note 1.—This result might be written $\pm x = \pm 2$; but this would not be different from $x = \pm 2$.

Note 2.—If we have $x^2 = a$, and a is not a perfect square, we write $x = \pm \sqrt{a}$.

Ex. 2. $\dfrac{x}{3} - \dfrac{x^2}{4} + \dfrac{x^2}{12} = \dfrac{8}{3}$.

Clearing the equation of fractions,

$$4x^2 - 3x^2 + x^2 = 32,$$

$$2x^2 = 32,$$

$$x^2 = 16,$$

$$x = \pm 4.$$

Examples—37.

Solve the following pure quadratics :

1. $9x^2 - 4 = 3x^2 + 5$.

2. $2x^2 - 12 = 36 - 8x^2$.

3. $\dfrac{4x^2}{9} = 625$.

4. $\dfrac{336}{7x^2} = 12$.

5. $\dfrac{x+1}{x-1} + \dfrac{x-1}{x+1} = \dfrac{13}{6}$.

6. $\dfrac{x^2-1}{7} - \dfrac{x^2}{9} = \dfrac{2x^2}{3} - 23$.

7. $(3x + 1)(3x - 1) = 81x^2 - 33$.

8. $\dfrac{x-2}{2} = \dfrac{30}{x+2}$.

9. $(x+2)^2 = 4x + 20$.

10. $x^2 - \dfrac{3x^2 - 4}{4} = 6 - \dfrac{6x^2 + 4}{7}$.

SECTION XXVIII.

SOLUTION OF AFFECTED QUADRATICS.

98. To solve an equation which has both x^2 and x in it, we bring it to a simple equation by taking the square root. To do this we have the following

Rule.—1. *Reduce the equation to the form* $x^2 + px = q$, *in which* x^2 *has* $+1$ *for coefficient.*

2. *Then add the square of half the coefficient of* x *to both sides; thus making the first side a perfect square, and preserving the equality.*

3. *Take the square root of both sides, putting the sign* $\pm$ *before the root of the second side. Then find* x *in this simple equation.*

Ex. 1. $3x^2 - 12x = 36$.

Dividing by 3, we have

$$x^2 - 4x = 12.$$

Adding the square of 2 to both sides, we get

$$x^2 - 4x + 4 = 12 + 4,$$

or $$x^2 - 4x + 4 = 16.$$

Taking the square root of both sides,

$$x - 2 = \pm 4;$$

whence, $$x = 2 + 4 = 6,$$

or $$x = 2 - 4 = -2.$$

Note.—When the second side is not a perfect square, we put the sign $\pm \sqrt{}$ over it.

Ex. 2. $x^2 + 6x = 2.$

Adding the square of 3 to both sides, we get

$$x^2 + 6x + 9 = 11.$$

Taking the square root, $$x + 3 = \pm \sqrt{11},$$

whence, $$x = -3 + \sqrt{11},$$

or $$x = -3 - \sqrt{11},$$

and we can only find x approximately by getting the approximate square root of 11.

Ex. 3. $3x^2 + 15x = 18.$

Dividing by 3, $$x^2 + 5x = 6,$$

$$x^2 + 5x + \tfrac{25}{4} = 6 + \tfrac{25}{4} = \tfrac{49}{4}$$

$$x + \tfrac{5}{2} = \pm \tfrac{7}{2}.$$

$$x = -\tfrac{5}{2} + \tfrac{7}{2} = 1, \quad \text{or} \quad x = -\tfrac{5}{2} - \tfrac{7}{2} = -6.$$

Ex. 4. $x^2 + \frac{10}{3}x = \frac{13}{3}$.

$$x^2 + \tfrac{10}{3}x + (\tfrac{5}{3})^2 = \tfrac{13}{3} + \tfrac{25}{9} = \tfrac{64}{9},$$

$$x + \tfrac{5}{3} = \pm\,\tfrac{8}{3},$$

and $$x = -\tfrac{5}{3} + \tfrac{8}{3} = -1, \quad \text{or} \quad x = -\tfrac{5}{3} - \tfrac{8}{3} = -\tfrac{13}{3}.$$

Ex. 5. $x^2 - \frac{1}{3}x = 34$.

$$x^2 - \tfrac{1}{3}x + (\tfrac{1}{6})^2 = 34 + \tfrac{1}{36} = \tfrac{1225}{36},$$

$$x - \tfrac{1}{6} = \pm\,\tfrac{35}{6},$$

and $$x = \tfrac{1}{6} + \tfrac{35}{6} = 6, \quad \text{or} \quad x = \tfrac{1}{6} - \tfrac{35}{6} = -\tfrac{17}{3}.$$

Examples—38.

1. $2x^2 - 12x + 40 = x^2 + 6x - 5$.

2. $\dfrac{x+2}{x-2} - \dfrac{x-2}{x+2} = 2\tfrac{2}{3}$.

3. $x^2 - 4x = 5$.

4. $x^2 - 3x = 2x^2 - x - \frac{1}{2}$.

5. $x^2 - 6x = -55$.

6. $x^2 - x = 42$.

7. $x^2 = \frac{3}{5}x - \frac{2}{25}$.

8. $2x^2 - 9x = 110$.

9. $100x^2 + 80x = 9$.

10. $9x^2 - 7x = 16$.

11. $11x^2 - 3x = 14$.

12. $\dfrac{1}{x+1} = \dfrac{2}{x} - \dfrac{2}{3}$.

13. $\dfrac{40}{x+1} + \dfrac{40}{x-1} = 9$.

14. $\dfrac{1}{x-2} - \dfrac{1}{x+3} = \dfrac{5}{36}$.

15. $\dfrac{3x+2}{3x-2} - \dfrac{3x-2}{3x+2} = 1\frac{1}{2}$.

16. $\dfrac{5+x}{5-x} - \dfrac{2x-4}{6-x} = 7$.

17. $(x-2)^2 + (x-3)(x-1) = x^2 - 8$.

SECTION XXIX.

PROBLEMS GIVING RISE TO QUADRATIC EQUATIONS.

99. Ex. 1. Find a number such that its square is equal to twice the product of two numbers, one of which is greater by 3, and the other less by 4.

Let $x =$ the number. Then $x + 3 =$ the number greater by 3, and $x - 4 =$ the number less by 4.

Then	$x^2 = 2(x+3)(x-4)$,
that is,	$x^2 = 2x^2 - 2x - 24$,
reducing,	$x^2 - 2x = 24$,
completing the square,	$x^2 - 2x + 1 = 24 + 1 = 25$,
taking the square root,	$x - 1 = \pm 5$,
whence	$x = 1 + 5 = 6$, or $x = 1 - 5 = -4$.

Examples—39.

1. Find a number which multiplied by its excess over 21 gives 196.

2. Find the number whose square increased by 4 times the number is equal to 117.

3. Find the number, 5 times the square of which divided by 3 is 135.

4. Find the number whose increase by 60 multiplied by its excess over 60 gives 6400.

5. Two numbers are in the ratio of 5 to 3, and the difference of their squares is equal to 144. Find the numbers.

6. Find the fraction which exceeds its square by $\frac{2}{9}$.

7. Find two numbers the difference of which is 5, and the product of the greater by their sum is equal to 150.

8. Find two consecutive numbers the product of which is 5 times the sum of the numbers increased by 5.

9. Find a number such that 12 divided by the number, added to 12 divided by the number increased by 9, is equal to 5.

10. A man bought \$100 worth of sheep. He lost 5 of them, and sold the rest for \$100, and gained \$1 a head on those sold. Find the number of sheep which he bought.

SECTION XXX.

EASY SIMULTANEOUS EQUATIONS SOLVED BY QUADRATICS.

100. We will consider the case of two equations and two unknowns, when one of the equations is of the first degree, or a simple equation, and the other of the second degree, or a quadratic equation. For this case we have the

Rule.—*From the equation of the first degree find the expression of one of the unknown quantities in terms of the other, and then substitute this expression in the second equation.*

Ex. 1. Given $\left.\begin{array}{ll} x + y = 10 & (1) \\ x^2 + y^2 = 52 & (2) \end{array}\right\}$ to find the values of x and y.

From (1), $y = 10 - x$.

Putting this in (2), $x^2 + (10 - x)^2 = 52$.

Expanding $(10 - x)^2$, $x^2 + 100 - 20x + x^2 = 52$.

Uniting terms, etc., $2x^2 - 20x = -48$.

Dividing by 2, $x^2 - 10x = -24$.

Completing square, $x^2 - 10x + 25 = -24 + 25 = 1$.

Taking the square root, $x - 5 = \pm 1$.

Transposing, $x = 6$ or 4.

Substituting value of x in (1), $y = 4$ or 6.

Ex. 2. $\left.\begin{array}{ll} 5x - 2y = 4 & (1) \\ 3xy - 4x^2 = 2 & (2) \end{array}\right\}$ to find the values of x and y.

From (1), $x = \dfrac{4 + 2y}{5}$.

Putting this for x in (2), $3y\left(\dfrac{4+2y}{5}\right) - 4\left(\dfrac{4+2y}{5}\right)^2 = 2$,

or $$\frac{12y + 6y^2}{5} - \frac{(64 + 64y + 16y^2)}{25} = 2.$$

Reducing, $$7y^2 - 2y = 57,$$

or $$y^2 - \frac{2y}{7} = \frac{57}{7},$$

and $$y = 3, \text{ or } -\frac{19}{7}.$$

The first value of y, substituted in first equation, gives

$$x = 2.$$

Examples—40.

Solve the simultaneous equations

$$\left.\begin{array}{r} x + y = 5 \\ xy = 6 \end{array}\right\} (1).$$

$$\left.\begin{array}{r} x + y = 11 \\ x^2 - y^2 = 11 \end{array}\right\} (2).$$

$$\left.\begin{array}{r} x^2 + y^2 = 100 \\ x + y = 14 \end{array}\right\} (3).$$

$$\left.\begin{aligned} x - y &= 5 \\ 3x + 4y &= 2xy - 12 \end{aligned}\right\} (4).$$

$$\left.\begin{aligned} xy &= 432 \\ \frac{y}{x} &= 3 \end{aligned}\right\} (5).$$

$$\left.\begin{aligned} xy &= 480 \\ 5x &= 6y \end{aligned}\right\} (6).$$

$$\left.\begin{aligned} \frac{x+y}{x} &= \frac{5}{3} \\ xy &= 6 \end{aligned}\right\} (7).$$

$$\left.\begin{aligned} x^2 - y^2 &= 57 \\ x - y &= 3 \end{aligned}\right\} (8).$$

$$\left.\begin{aligned} 2x + 6y &= 38 \\ 6x^2 - 2y^2 &= 46 \end{aligned}\right\} (9).$$

$$\left.\begin{aligned} xy + 7y &= 24 \\ x - y &= 3 \end{aligned}\right\} (10).$$

$$\left.\begin{aligned} x + y &= 7 \\ 2x + 7y^2 &= 4xy - 2 \end{aligned}\right\} (11).$$

SECTION XXXI.

RADICALS OF THE SECOND DEGREE.

101. Radicals or Surds.—The indicated roots of quantities which are not perfect powers are called Radical expressions, or simply *Radicals* or *Surds*, as, for example, $\sqrt{5a}$, $\sqrt[3]{11}$, $\sqrt[4]{8}$, $3\sqrt[6]{12}$, which interpreted by signs mean that $\sqrt{5a} \times \sqrt{5a} = 5a$; $\sqrt[3]{11} \times \sqrt[3]{11} \times \sqrt[3]{11} = 11$; $\sqrt[4]{8} \times \sqrt[4]{8} \times \sqrt[4]{8} \times \sqrt[4]{8} = 8$, etc.

102. Radicals of the Second Degree.—The indicated square roots of quantities which are not perfect squares are called radicals of the second degree, as $5\sqrt{a}$, $\sqrt{7a^2}$, $\sqrt{8ab}$, etc.

103. Coefficients.—The factor before the radical sign is called the coefficient of the radical. Thus in the radicals $5\sqrt{a}$, $6a\sqrt{bc}$, 5 and $6a$ are the coefficients respectively. When no coefficient is written, 1 is understood.

104. 1. *A coefficient of a radical of the second degree may be passed under the radical sign as a factor by squaring it.* For, since the $\sqrt{(5)^2}$ is 5, then $5\sqrt{a} = \sqrt{5^2 \times a} = \sqrt{25a}$.

2. *Any perfect square factor of an expression under the radical sign may be transferred as a factor before the sign by taking its square root.*

Thus, $\sqrt{25a^2b} = \sqrt{(5a)^2 \times b} = 5a\sqrt{b}$,

$$7\sqrt{45} = 7\sqrt{9 \times 5} = 7 \times 3\sqrt{5} = 21\sqrt{5}.$$

105. Simplest Form of Radicals.—A radical quantity of the second degree is in its *simplest form* when the number or expression under the radical sign contains no factor greater than 1 which is a perfect square.

Thus, $5\sqrt{3}$ is in its simplest form, but $2\sqrt{8a^2}$ is not in its simplest form, since it may be written $2\sqrt{4a^2 \times 2}$, and taking the square root of the factor $4a^2$, we have $2\sqrt{8a^2} = 2\sqrt{4a^2 \times 2} = 4a\sqrt{2}$.

106. Reduction of Radicals.—To reduce a radical of the second degree to its simplest form, we have the following

Rule.—*Separate the part under the radical sign into two factors, one of which contains all the factors which are perfect squares. Take the square root of this factor and multiply it by the coefficient of the radical, leaving the other factor under the radical sign.*

Ex. 1. Reduce $\sqrt{54}$ to its simplest form.

Reduction, $\sqrt{54} = \sqrt{9 \times 6} = 3\sqrt{6}$.

Ex. 2. Reduce $2\sqrt{27a^2b^2c}$ to its simplest form.

Reduction, $2\sqrt{27a^2b^2c} = 2\sqrt{9a^2b^2 \times 3ac} = 2 \times 3ab\sqrt{3ac} = 6ab\sqrt{3ac}$.

107. Similar Radicals.—Radicals of the second degree are said to be *similar* when they have the same quantities under the radical sign. Thus $5\sqrt{3}$, $8\sqrt{3}$, $10\sqrt{3}$ are similar radicals, and $\sqrt{12}$ may be rendered similar to all these because in its simplest form $\sqrt{12} = 2\sqrt{3}$.

108. Addition and Subtraction of Radicals.—To add or subtract similar radicals,

Rule.—*Add or subtract their coefficients, and place the result as a coefficient before the common radical.*

Ex. 1. $6\sqrt{a} + 12\sqrt{a} + 23\sqrt{a} = 41\sqrt{a}$.

Ex. 2. $4\sqrt{5a} - 3\sqrt{5a} = \sqrt{5a}$.

Ex. 3. Add $3\sqrt{54}$, $5\sqrt{24}$, and $3\sqrt{6}$.

Here $3\sqrt{54} = 3\sqrt{9 \times 6} = 9\sqrt{6}$, $5\sqrt{24} = 5\sqrt{4 \times 6} = 10\sqrt{6}$,

and $9\sqrt{6} + 10\sqrt{6} + 3\sqrt{6} = 22\sqrt{6}$.

109. Multiplication and Division of Radicals.—To multiply or divide two radicals of the second degree,

Rule.—*Multiply or divide the coefficients for the new coefficient, and the parts under the radical for the new radical factor.*

Ex. 1. Multiply $5\sqrt{7}$ by $8\sqrt{3}$. Result, $40\sqrt{21}$.

Ex. 2. Divide $9a^2\sqrt{2b}$ by $3a\sqrt{2}$. Result $3a\sqrt{b}$.

Ex. 3. Square $3 + \sqrt{5}$.

$$\begin{array}{rl}
\text{Process,} & 3 + \sqrt{5} \\
 & 3 + \sqrt{5} \\
\hline
 & 9 + 3\sqrt{5} \\
 & \quad\; 3\sqrt{5} + 5 \\
\hline
\text{Result,} & 9 + 6\sqrt{5} + 5
\end{array}$$

Ex. 4. Multiply $3 + \sqrt{2}$ by $3 - \sqrt{2}$.

$$(3 + \sqrt{2})\,(3 - \sqrt{2}) = (3)^2 - (\sqrt{2})^2 = 9 - 2 = 7.$$

NOTE.—The above rules for the addition, subtraction, multiplication, and division of radicals of the second degree apply to all radicals of the *same index or degree.*

110. Radical Expressions of the Form $\sqrt{\frac{a}{b}}$. Fractional radical expressions of the form $\sqrt{\frac{a}{b}}$, are simplified by the following

Rule.—*Multiply the numerator and denominator of the fraction by any number which will make the denominator a perfect square;* then apply the Rule of Art. 104.

$$\text{Thus } \sqrt{\frac{a}{b}} = \sqrt{\frac{ab}{b^2}} = \sqrt{\frac{1}{b^2} \times ab} = \frac{1}{b}\sqrt{ab}.$$

Ex. Reduce $5\sqrt{\frac{2}{3}}$ to its simplest form.

$$5\sqrt{\tfrac{2}{3}} = 5\sqrt{\tfrac{6}{9}} = 5\sqrt{\tfrac{1}{9} \times 6} = \tfrac{5}{3}\sqrt{6}.$$

111. Radical Expressions of the Form $\frac{c}{a \pm \sqrt{b}}$. In order to remove the radical quantity from the denominator of such expressions as $\frac{c}{a + \sqrt{b}}$ or $\frac{c}{a - \sqrt{b}}$, we multiply the numerator and denominator of the first by $a - \sqrt{b}$, and of the second by $a + \sqrt{b}$.

Thus $$\frac{c}{a+\sqrt{b}} = \frac{c(a-\sqrt{b})}{(a+\sqrt{b})\,(a-\sqrt{b})} = \frac{ac - c\sqrt{b}}{a^2 - b};$$

and $$\frac{c}{a-\sqrt{b}} = \frac{c(a+\sqrt{b})}{(a-\sqrt{b})\,(a+\sqrt{b})} = \frac{ac + c\sqrt{b}}{a^2 - b}.$$

Similarly $$\frac{1}{2+\sqrt{3}} = \frac{2-\sqrt{3}}{(2+\sqrt{3})\,(2-\sqrt{3})} = \frac{2-\sqrt{3}}{4-3} = 2 - \sqrt{3}.$$

Examples—41.

1. Simplify $\sqrt{28a^3b^2c^6}$.
2. Simplify $\sqrt{x^3 - ax^2}$.
3. Reduce $\sqrt{27}$, $\sqrt{12}$, and $5\sqrt{48}$ to similar radicals.
4. Reduce $\sqrt{8}$, $\sqrt{50}$, and $\sqrt{72}$ to similar radicals.
5. Add $\sqrt{2}$, $5\sqrt{8}$, $4\sqrt{18}$, $3\sqrt{128}$.
6. Multiply $3\sqrt{2} - 5\sqrt{3}$ by $2\sqrt{2}$.
7. Multiply $2 + \sqrt{3}$ by $3 - \sqrt{3}$.
8. Multiply $\sqrt{a+b}$ by $\sqrt{a-b}$.
9. Multiply $\sqrt{5} + \sqrt{3}$ by $\sqrt{5} - \sqrt{3}$.
10. Square $2 - \sqrt{3}$.
11. Simplify $\sqrt{\frac{3}{7}}$.
12. Simplify $\frac{3}{4}\sqrt{\frac{1}{2}}$.
13. Divide $2\sqrt{3}$ by $3\sqrt{5}$ and simplify the result.

14. Remove the radical from the denominator of $\dfrac{3}{\sqrt{2}-1}$.

15. Remove the radical from the denominator of $\dfrac{2}{\sqrt{3}+1}$.

16. Remove the radicals from the denominator of $\dfrac{1}{\sqrt{5}-\sqrt{2}}$.

17. Simplify $\sqrt{80} + 3\sqrt{20} - 4\sqrt{45} + 2\sqrt{5}$.

EQUATIONS CONTAINING RADICALS OF THE SECOND DEGREE.

112. When the unknown quantity in an equation is under a radical sign, it becomes necessary in solving it to *clear the equation of radicals.* If the radicals are of the second degree, and if there be only one in the equation, we have the following

Rule.—*Transpose the terms so that the radical shall be on one side of the equation and all the other terms on the other side. Then square both sides.*

EXAMPLE.—Given $\sqrt{6+x} + 8 = 14$, to find x.

Transposing, $\sqrt{6+x} = 6$.

Squaring both sides, $6 + x = 36$;

and $x = 30$.

113. If the equation contains two radicals of the second degree with the unknown letter under them, *we must repeat the above operation for the second radical;* that is, *two* transpositions and *two* squarings will be necessary.

EXAMPLE.—Given $\sqrt{x+6} + \sqrt{x-5} = 11$, to find x.

Transposing, $\sqrt{x+6} = 11 - \sqrt{x-5}$.

Squaring, $x + 6 = (11 - \sqrt{x-5})^2$;

or $x + 6 = 121 - 22\sqrt{x-5} + x - 5$.

Transposing and reducing,

$$22\sqrt{x-5} = 110;$$

or $\sqrt{x-5} = 5$.

Squaring again, $x - 5 = 25$;

and $x = 30$.

Examples—42.

Find x in the following equations:

1. $\sqrt{a+x} + b = c$.
2. $\sqrt{x+25} - \sqrt{x} = 1$.
3. $\sqrt{x+13} - \sqrt{x-11} = 2$.
4. $\sqrt{x+4} + \sqrt{x-1} = 5$.
5. $\sqrt{2x+14} + \sqrt{2x-14} = 14$.
6. $\sqrt{4x+4} + 9 = 13$.
7. $\sqrt{x+4} + \sqrt{x-4} = 4$.
8. $\sqrt{5x-1} + \sqrt{10x+5} = 8$.
9. $\sqrt{x+9} = 13 - 2\sqrt{x}$.

B

SECTION XXXII.

RATIO AND PROPORTION.

114. **Ratio** is the relation which one quantity bears to another in respect of magnitude, and is measured *by the number of times the one is contained in the other*, or *by the part or parts the one is of the other.*

Thus, the ratio of 10 to 5 is 2, and the ratio of 5 to 10 is $\frac{5}{10}$, or $\frac{1}{2}$, as 5 is one-half of 10.

115. Hence the fraction $\frac{a}{b}$ represents the ratio of a to b. This ratio is written $a:b$. Hence, $a:b=\frac{a}{b}$, and similarly $c:d=\frac{c}{d}$, etc.

116. **Proportion.**—If $\frac{a}{b}=\frac{c}{d}$, $a:b=c:d$, and this equality of two ratios is called a *Proportion.* It is usually written $a:b::c:d$, and is read "a is to b as c is to d."

The first and third terms of the proportion are called the *antecedents*, and the second and fourth the *consequents.* Again, the first and fourth terms are called the *extremes*, and the second and third the *means.*

117. **Results to be Remembered.**—We must bear in mind:

1. The value of any ratio $a:b$ is the fraction $\frac{a}{b}$.

2. If $a:b::c:d$, then $\frac{a}{b}=\frac{c}{d}$.

3. Then, the rules given for fractions and equations apply to ratios and proportions.

118. If the terms of a ratio be multiplied or divided by the same number the ratio is not changed. For $\frac{a}{b} = \frac{ma}{mb}$, and therefore $a : b :: ma : mb$.

Examples—43.

Find the values of the following ratios:

1. $3x : 21x$; $am : mn$.
2. $2a^2 : 8a^4$; $cxy : 3y$.
3. $ac : bc$; $5acx : 4a^2x$.
4. $bcx^2 : 5acx$; $4a^2c^2 : 12a^2c^3$.
5. $ax + bx : 2x^2$; $2ab + b^2 : bc$.
6. $1 - a^2 : 1 + a$; $a^2 - x^2 : a - x$.
7. $\frac{1}{2}bc : \frac{2}{3}ac$; $\frac{5}{6}amn : \frac{7}{24}an^2$.
8. Which is the greater $16 : 17$, or $17 : 18$.

PROPORTION.

119. *In any proportion the product of the extremes is equal to the product of the means.*

For let $a : b :: c : d$ be the given proportion.

Then $\frac{a}{b} = \frac{c}{d}$, and clearing of fractions, $ad = bc$, which was to be proved.

It follows that if $a : b :: b : c$, then $b^2 = ac$, or $b = \sqrt{ac}$; b is then said *to be a mean proportional between* a *and* c.

Hence, **Rule.**—*The mean proportional between two numbers is the square root of their product.*

120. Conversely, *if* ad = bc, *then the proportion* a : b :: c : d, *is true.*

For, dividing both sides of the equation $ad = bc$ by bd, we have $\frac{a}{b} = \frac{c}{d}$. Hence (Art. 117), $a : b :: c : d$.

121. Proof of the Single Rule of Three.—It follows from the above, if three terms of a proportion are given, the fourth may be found.

For, let the given terms be a, b, c, in order, and the unknown term be x; then $a : b :: c : x$, and therefore $ax = bc$, or $x = \frac{bc}{a}$. Hence the rule as given in the arithmetic.

122. If $a : b :: c : d$, then $b : a :: d : c$.

For, since $a : b :: c : d$, then $bc = ad$ (Art. 119); and dividing both sides by ac, we have $\frac{b}{a} = \frac{d}{c}$, or

$$b : a :: d : c.$$

123. If $a : b :: c : d$, then $a : c :: b : d$.

For, $ad = bc$ (Art. 119); and dividing both sides of this equation by dc, we have $\frac{a}{c} = \frac{b}{d}$, or

$$a : c :: b : d.$$

124. If $a : b :: c : d$, then $a + b : b :: c + d : d$.

For, first, $\frac{a}{b} = \frac{c}{d}$, and adding 1 to both sides of this equation, we have $\frac{a}{b} + 1 = \frac{c}{d} + 1$.

Hence, $$\frac{a+b}{b} = \frac{c+d}{d};$$

$$\therefore\quad a+b : b :: c+d : d.$$

Also, since $\dfrac{b}{a} = \dfrac{d}{c}$, it is clear that

$$a+b : a :: c+d : c.$$

125. If $a : b :: c : d$, then $a-b : b :: c-d : d$.

First, $\dfrac{a}{b} = \dfrac{c}{d}$, and subtracting 1 from both sides of this equation, we have

$$\frac{a}{b} - 1 = \frac{c}{d} - 1,$$

or $$\frac{a-b}{b} = \frac{c-d}{d}.$$

Hence, $$a-b : b :: c-d : d.$$

Also, $$a-b : a :: c-d : c.$$

126. If $a : b :: c : d$ (1), and $a : b :: e : f$ (2), then $c : d :: e : f$.

For, first, $$\frac{a}{b} = \frac{c}{d},$$

and second, $$\frac{c}{d} = \frac{e}{f}.$$

Hence, $$\frac{a}{b} = \frac{e}{f}.$$

Therefore, $$a : b :: e : f.$$

Examples—44.

1. If $a : b :: c : d$, show that $5a : 6b :: 5c : 6d$.

2. Change $5 : x + 3 :: x - 3 : 8$ into an equation, and find x.

3. Change $4x = 5y$ into a proportion.

4. If $2 - x : 2 + x :: 5 : 6$, find the ratio $2 : x$.

5. The first, second, and fourth terms of a proportion are ax, $3cx$, and $\dfrac{12bcm}{a}$, what is the third term?

6. There are two numbers in the ratio of 2 to 3, and if each of these be increased by 2 the ratio is then 5 to 7. Find the two numbers.

7. If $6a - 4c = 4d - 6b$, show that $a + b : c + d :: 2 : 3$.

8. Find two numbers in the ratio of 4 to 5, the difference of which bears to 54 the ratio of 1 to 9.

9. Given $x + y : x - y :: 6 : 5$. Find the ratio $x : y$.

10. Find the mean proportional between $16 - x^4$ and $\dfrac{4 + x^2}{4 - x^2}$.

11. Change $x + 4 : 4 :: 12 : x + 6$ into an equation, and find x.

12. There are four consecutive numbers, the last of which bears to the first the ratio of 8 to 7. Find them.

SECTION XXXIII.

ARITHMETICAL PROGRESSION.

127. An Arithmetical Progression is a series of numbers which go on increasing or decreasing by a fixed number called the *Common Difference.*

Thus 1, 3, 5, 7, 9, 11, etc., form an arithmetical progression with the common difference 2.

So also 20, 17, 14, 11, 8, etc., is an arithmetical progression with the common difference 3.

Again, a, $4a$, $7a$, $10a$, etc., is an arithmetical progression with the common difference $3a$.

128. General Forms of Arithmetical Progressions.—If a be the first term of an arithmetical progression, and d the common difference, then the general algebraic forms will be

1. a, $a + d$, $a + 2d$, $a + 3d$, etc., for an increasing arithmetical progression.

2. a, $a - d$, $a - 2d$, $a - 3d$, etc., for a decreasing arithmetical progression.

129. To find any Term.—From these we see that given the first term a, and common difference d, of an arithmetical progression, we can find any term.

Thus (1.) in an increasing arithmetical progression the 6th term is $a + 5d$, the 10th term is $a + 9d$, and the nth term is $a + (n - 1)d$.

(2.) In a decreasing arithmetical progression the nth term is $a - (n - 1)d$.

Therefore to find any term of an arithmetical progression, we have the

Rule.—*Multiply the common difference by the number of terms which precede the required term, and add this product to the first term for an increasing progression, and subtract it from the first term for a decreasing progression.*

130. If l be the nth term, this rule may be expressed briefly as follows:

$l = a + (n - 1)d$, for an increasing arithmetical progression.

$l = a - (n - 1)d$, for a decreasing arithmetical progression.

Ex. 1. Find the 25th term of the arithmetical progression 3, 5, 7, 9, etc.

Here $a = 3$, and $d = 2$, and $n = 25$.

Hence $l = 3 + (25 - 1)2 = 3 + 48 = 51$.

Ex. 2. Find the 20th term of the arithmetical progression 80, 76, 72, 68, etc.

Here $a = 80$, $d = 4$, and $n = 20$.

Hence $l = 80 - (20 - 1)4 = 80 - 76 = 4$.

131. Sum of the Terms.—To find a rule for the sum of n terms of an arithmetical progression, we write first

$$S = a + a + d + a + 2d + a + 3d, \text{ etc., to } n \text{ terms};$$

and also, $$\underline{S = l + l - d + l - 2d + l - 3d, \text{ etc., to } n \text{ terms.}}$$

Adding, we have $2S = (a + l) + (a + l) + (a + l) + (a + l)$ to n terms,

or $$2S = (a + l)n.$$

Therefore $$S = \left(\frac{a + l}{2}\right)n.$$

Hence, **Rule.**—*Multiply half the sum of the first and last terms by the number of terms.*

Example.—Find the sum of 30 terms of the arithmetical progression 1, 3, 5, 7, 9, etc.

Here $$a = 1, \quad d = 2, \quad \text{and} \quad n = 30.$$

Hence, first, $l = 1 + (30 - 1)\,2 = 1 + 58 = 59$,

and, second, $$S = \frac{1 + 59}{2} \times 30 = 30 \times 30 = 900.$$

132. Arithmetical Means.—Numbers inserted between two given numbers, and forming an arithmetical progression with the given numbers as first and last terms, are called arithmetical means.

To find a rule for inserting any number of arithmetical means between two numbers a and b, we have only to find

the common difference. To do this, we take the formula, Art. 130,

$$l = a + (n - 1)d,$$

whence $$d = \frac{l - a}{n - 1}.$$

Now if m be the number of means to be inserted, between a and l, then $n = m + 2$, and $n - 1 = m + 1$.

Therefore $d = \dfrac{l - a}{m + 1}$, which gives

Rule.—*To find the required common difference, divide the difference between the numbers by the number of means to be inserted plus* 1.

Ex. 1. Insert 4 arithmetical means between 5 and 20.

Here the common difference $d = \dfrac{20 - 5}{4 + 1} = \dfrac{15}{5} = 3.$

Hence the four required means are 8, 11, 14, 17, and the arithmetical progression is 5, 8, 11, 14, 17, 20.

133. To find an arithmetical mean between two numbers a and l, or their *average*, let $x =$ this average or arithmetical mean, and d the common difference.

Then $$x = a + d,$$

also, $$x = l - d,$$

and adding, $$2x = a + l,$$

or $$x = \frac{a + l}{2}.$$

Hence, **Rule.**—*The arithmetical mean or average of two numbers is half their sum.*

Examples—45.

1. Find the 14th and 28th terms in the progressions:

(1.) 3, 7, 11, etc.

(2.) 5, 8, 11, 14, etc.

(3.) $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, etc.

(4.) 200, 195, 190, etc.

2. Find the sum of 20 terms of the progressions:

(1.) 1, 4, 7, 10, etc.

(2.) 100, 96, 92, etc.

(3.) 15, $14\frac{2}{3}$, $14\frac{1}{3}$, etc.

3. The clocks of Venice strike the hours from 1 to 24. Find the number of strokes in a day.

4. A man employed a workman for 90 days at 1 cent the first day, 3 cents the next, 5 cents the next, etc. Find the whole amount of the workman's wages for the 90 days.

5. In a potato race each man picked up 50 potatoes placed two feet apart in a line, and put them in a basket in the line at two feet from the first potato. How far did each run, starting at a basket?

6. Show that the sum of 50 terms of the arithmetical progression 1, 3, 5, 7, 9, etc., is 50^2, and the sum of n terms is n^2.

7. Show that the sum of the first n natural numbers 1, 2, 3, 4, 5, 6, 7, 8, etc., is $\frac{n^2 + n}{2}$.

8. Find the arithmetical mean of $\frac{1}{3}$ and $\frac{1}{5}$.

9. Find the arithmetical mean of $a + b$ and $a - b$.

10. Insert two arithmetical means between 10 and 11.

11. Insert 4 arithmetical means between 50 and 40.

12. Insert 20 arithmetical means between $3\frac{1}{3}$ and $3\frac{2}{3}$.

13. A heavy body falling from a height falls $16\frac{1}{12}$ feet the first second of its fall, and in each succeeding second $32\frac{1}{6}$ more feet than in the preceding (the resistance of the air being left out). How far would a body fall in 20 seconds?

SECTION XXXIV.

GEOMETRICAL PROGRESSION.

134. A **Geometrical Progression** is a series of numbers which go on increasing or decreasing in the same *fixed ratio,* that is, by a common multiplier.

Thus, 2, 4, 8, 16, etc., are in geometrical progression, each term being twice the preceding term.

Also, 81, 27, 9, 3, 1, are in geometrical progression, each term being $\frac{1}{3}$ of the preceding term.

135. This common multiplier is called the *Common Ratio,* and may be whole or fractional. It is found in any geometrical progression by *dividing any term by the preceding term.*

Ex. 1. The common ratio in the geometrical progression 3, 9, 27, 81, etc., is $\frac{9}{3} = 3$.

Ex. 2. The common ratio in the geometrical progression 64, 16, 4, 1, is $\frac{1}{4}$.

Ex. 3. Are $\frac{1}{4}$, $\frac{1}{5}$, and $\frac{4}{25}$ in geometrical progression? And if so, what is the common ratio?

Ans. $\frac{1}{5} \div \frac{1}{4} = \frac{4}{5}$, and $\frac{4}{25} \div \frac{1}{5} = \frac{4}{5}$. Hence the given factors are in geometrical progression, and $\frac{4}{5}$ is the common ratio.

136. General Form of a Geometrical Progression.—If a is the first term of a geometrical progression, and r the common ratio,

The second term is ar,

The third term is ar^2,

The fourth term ar^3,

and so on, the exponent of r, in any term, being always 1 less than the number of the terms. Therefore, the nth term will be $l = ar^{n-1}$, and the series expressed in algebraic form will be

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4 \;\cdots\cdots\; ar^{n-1}.$$

137. To Find any Term.—The expression $l = ar^{n-1}$, gives us, for finding any term of a geometrical progression when the first term and common ratio are known, the following

Rule.—*Multiply the first term by the ratio raised to a power one less than the number of terms.*

EXAMPLE.—Find the 10th term of the geometrical progression 2, 4, 8, 16, etc.

$$l = 2 \times 2^9 = 1024.$$

138. The Sum of the Terms of a Geometrical Progression.—Let S be the sum of the terms of a geometrical progression $a,\ b,\ c,\ d,\ e, f \cdot \cdot \cdot \cdot\ k,\ l,$ and r the common ratio, then

$$\begin{aligned} b &= ar, \\ c &= br, \\ d &= cr, \\ &\cdot\ \cdot\ \cdot\ \cdot \\ l &= kr, \end{aligned}$$

or adding, $b + c + d \cdot \cdot \cdot + l = (a + b + c + \cdot \cdot \cdot + k)r.$

But the first side has all the terms in it except a, and the brackets, on the second side, contain all the terms except l.

Hence,

$$S - a = (S - l)r,$$

or

$$S - a = Sr - lr.$$

Therefore,

$$S = \frac{lr - a}{r - 1}.$$

This expression gives S when r, l, and a are known. If the geometrical progression is a decreasing one, then the value of S is written $S = \dfrac{a - lr}{r - 1}$.

EXAMPLE.—Find the sum of 11 terms of the geometrical progression 2, 4, 8, etc.

Here $\qquad a = 2, \quad r = 2, \quad l = 2 \times 2^{10} = 2048.$

Hence, $\qquad S = \dfrac{2 \times 2048 - 1}{2 - 1} = 4094.$

139. In the expression $S = \dfrac{a - lr}{1 - r}$ for a decreasing progression, when the number of terms increases indefinitely (*i. e.*, becomes great without limit), lr becomes a smaller and smaller fraction, and if we neglect it we get $\dfrac{a}{1 - r}$ as a quantity greater than the sum of the series when the number of the terms is infinite.

We will call this Σ (sigma), *the limit of the sum* of the terms of a decreasing geometrical progression when the number of terms is great without limit, and write it

$$\Sigma = \frac{a}{1 - r}.$$

Ex. 1. Find the limit of the sum of the terms of the geometrical progression 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, etc., for an infinite number of terms.

$$\Sigma = \frac{a}{1 - r} = \frac{1}{1 - \frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2.$$

This result shows that the sum of all the numbers that can be written in the geometrical progression 1, $\frac{1}{2}$, $\frac{1}{4}$, etc., can never be as great as 2.

Ex. 2. Find the limit of the sum of the terms of the geometrical progression $\frac{3}{10}$, $\frac{3}{100}$, $\frac{3}{1000}$, $\frac{3}{10000}$, etc., or, in other terms, the recurring decimal .333333, etc.

Here $\qquad a = \frac{3}{10}$, and $r = \frac{1}{10}$.

Hence, $\qquad \Sigma = \dfrac{\frac{3}{10}}{1 - \frac{1}{10}} = \dfrac{\frac{3}{10}}{\frac{9}{10}} = \frac{3}{9} = \frac{1}{3}.$

140. EXAMPLE.—Insert a geometric mean between two numbers a and b. Let x be the mean. Then the common ratio $= \dfrac{x}{a}$, and also $= \dfrac{b}{x}$.

Therefore, $\dfrac{x}{a} = \dfrac{b}{x}$, or $x^2 = ab$;

and $x = \sqrt{ab}$. Same result as in Art. 119 in Proportion.

141. EXAMPLE.—Insert two geometric means between $\frac{1}{4}$ and 2. Let $x =$ the required common ratio.

Then the series will be

$$\tfrac{1}{4},\ \tfrac{1}{4}x,\ \tfrac{1}{4}x^2,\ 2.$$

Therefore the common ratio

$$x = \frac{2}{\frac{1}{4}x^2};$$

whence $x^3 = 8$.

Therefore, $x = 2$,

and the series is $\frac{1}{4},\ \frac{1}{2},\ 1,\ 2.$

Examples—46.

1. Find the common ratio in the following geometrical progressions:

(1.) 100, 300, 900, etc.
(2.) $3\frac{1}{2}$, 7, 14, etc.
(3.) $\frac{1}{3}$, $\frac{1}{2}$, $\frac{3}{4}$, etc.

(4.) .2, .02, .002, etc.

(5.) .625, 1.25, 2.5, etc.

(6.) a, $3ax$, $9ax^2$, etc.

2. Find the geometric mean between 50 and $12\frac{1}{2}$.

3. Insert two geometric means between 6 and 162.

4. Insert two geometric means between $\frac{1}{27}$ and 1.

5. Which is greater, the arithmetical mean or the geometric mean between 1 and $\frac{1}{25}$?

6. Find the sum of 10 terms of the geometrical progression 1, 3, 9, 27, etc.

7. A blacksmith used eight nails in putting a shoe upon a horse's foot; he received 1 cent for the first nail, 2 cents for the second, 4 for the third, and so on. What did he receive for the shoeing?

8. Find the limit of the sum of an infinite number of terms of the geometrical progression $\frac{1}{4}$, $\frac{1}{16}$, $\frac{1}{64}$, etc.

9. Find the limit of the value of the circulating decimal .55555 . . . . . .

10. Find the limit of the value of the decimal .212121 . . .

11. Insert three geometric means between 12 and 192.

SECTION XXXV.

MISCELLANEOUS EXAMPLES.

142. The following examples may be used according to the preference of the teacher, either for review and examination after the pupil has accomplished the preceding sections, or may be drawn on for additional practice while advancing from section to section.

(Sections I–VIII.)

1. Simplify $2a^2x - 3ax^2 + x^3 - 5a^2x + 6ax^2 - 3x^3$.

2. Subtract $3a^2b - 2a^3 + 6 - 8x$ from $-3a^2b + 2a^3 + 6 - 8x$.

3. Add $a - b + c$, $2a + 2b - 3c - 3a + 4b + 4c$, and subtract the sum from $4a + 5b - 5c$.

4. Multiply $2a - 3b$ by $2a + 3b$.

5. Multiply $3a + b$ by $3a - b$.

6. Multiply $a + b + c + d$ by $a + b - c - d$.

7. Multiply $a^2 + ab + b^2$ by $a^2 - ab + b^2$.

8. Write $(5a + 100b)^2$ and $(14a - 11b)^2$.

9. Divide $25a^3x^7y^2$ by $5ax^4y$, and the quotient by $5a^2x^3y$.

10. Divide $210a^4x^3 - 140a^3x^2$ by $35a^2x$.

11. Divide $81x^4 - y^4$ by $3x - y$.

12. Divide $a^6 - 1$ by $a^3 + 2a^2 + 2a + 1$.

13. Divide $x^4 - 4a^3x + 3a^4$ by $x^2 - 2ax + a^2$.

14. Simplify $(a - x) - (2x - b) - (2 - 2a) + (3 - 2x) - (1 - x)$.

15. Simplify $(x^3 - 2cx^2 + 3c^2x) - (cx^2 - 2x^3 + 2cx^2) + (x^3 - c^2x - cx^2)$.

16. Simplify $6(1 - x) + 2(1 + 6x)$.

17. Simplify $(1 + x)(1 - x)(1 + x^2)$.

18. Simplify $\frac{1}{2}(a + x) - \frac{1}{2}(a - x)$.

19. Find the factors of $a^4 - b^4$.

20. Find the factors of $x^2 - 11x + 30$.

21. Find the factors of $x^2 - x - 30$.

22. Find the factors of $4m^2 - 9n^2$.

23. Find the factors of $16a^2x^2 - 25y^2$.

24. Find the factors of $x^2 - 9x + 14$.

25. Find the factors of $x^2 - 6x - 7$.

26. Two factors of $b^3 - 7b + 6$ are $b - 1$ and $b - 2$. Find the other.

(Sections IX, X.)

27. Find the G. C. D. of $9a^2cx$, $2a^2x^2$, and $7acx$.

28. Find the G. C. D. of $a^2 - b^2$ and $(a - b)^2$.

29. Find the G. C. D. of $x^2 + x - 6$ and $x^2 + 5x + 6$.

30. Find the G. C. D. of $a^2 + 3a + 2$ and $a^2 + 4a + 3$.

31. Find the G. C. D. of $3a^2 + a - 2$ and $3a^2 + 4a - 4$.

32. Find the L. C. M. of $7x^5$, $21x^3$, $63x^7$.

33. Find the L. C. M. of $14a^3$, $63a^2x$, $28ax^2$, and $70x^3$.

34. Find the L. C. M. of $340ab^2c$ and $221a^2b^2c^2$.

35. Find the L. C. M. of $9(x+1)$ and $6(x+2)$.

36. Find the L. C. M. of $30(x-1)$ and $45(x+1)$.

37. Find the L. C. M. of $6(x+a)$, $12(x-a)$, and $18(x^2-a^2)$.

38. Find the L. C. M. of a^2+6a-7 and a^2+8a+7.

(Sections XI–XIV.)

39. Reduce $\dfrac{168a^3b^2c}{48a^2bc^3}$ to its lowest terms.

40. Reduce $\dfrac{2abc}{15a^2}$, $\dfrac{a^2-x^2}{a+x}$, $\dfrac{x^2+4x+3}{x^2+3x+2}$ each to lowest terms.

41. Add together $\dfrac{a}{x}$, $\dfrac{3a}{4x}$, $\dfrac{4a}{5x}$.

42. Add together $\dfrac{3a}{a-b}$, $\dfrac{x-3a}{a-b}$, $\dfrac{a}{a-b}$.

43. Add together $\dfrac{c}{a-x}$ and $\dfrac{c}{a+x}$.

44. Add together $\dfrac{1}{a+b}$, $\dfrac{2b}{a^2-b^2}$, $\dfrac{1}{a-b}$.

45. Add $\dfrac{1}{x+1}$, $\dfrac{1}{x-1}$, and $\dfrac{2x}{x^2-1}$.

46. From $\dfrac{x+8}{x-2}$ subtract $\dfrac{x-7}{x-2}$.

47. From $\dfrac{3}{x-1}$ take $\dfrac{3}{x+1}$.

48. From $\dfrac{a+1}{a-1}$ take $\dfrac{a-1}{a+1}$.

49. Simplify $\dfrac{7a-4}{3} - \dfrac{11a-7}{5}$.

50. Simplify $1 - \dfrac{2a}{1+a}$ and $1 + \dfrac{a}{1-a}$.

(Sections XV–XVII.)

51. Multiply $\dfrac{1}{a+1} + \dfrac{1}{a-1}$ by $\dfrac{1}{2a}$.

52. Multiply $\dfrac{2a-1}{8\frac{1}{2}}$ by 17.

53. Multiply $\dfrac{x^2+2x+1}{x^2-5x+6}$ by $\dfrac{x^2-3x+2}{x^2+4x+3}$.

54. Simplify $\dfrac{a^2}{bc} \times \dfrac{b^2}{ac} \times \dfrac{c^2}{ab}$.

55. Simplify $\dfrac{a+1}{a-1} \times \dfrac{a+2}{a^2-1} \times \dfrac{a-1}{(a+2)^2}$.

56. Divide $\dfrac{4a^2b}{5cx^2}$ by $\dfrac{2ab^2}{15c^2x}$.

57. Divide $\dfrac{1}{a^2-b^2}$ by $\dfrac{1}{a-b}$.

58. Divide $\dfrac{a}{2}$ by $\dfrac{1}{2} - \dfrac{a}{2}$.

59. Divide $\frac{1}{a+1}$ by $1 - \frac{1}{a+1}$.

60. Divide $a + 1$ by $\frac{a+1}{a} + 1$.

61. Divide $\frac{x^2 - 4x + 4}{x^2 - 6x + 9}$ by $\frac{x^2 - 3x + 2}{x^2 - 4x + 3}$.

62. Find the numerical value of $3a - 2b + 2c - (4b - (bc - bd))$ when $a = 4$, $b = 1$, $c = -1$, $d = 0$.

63. Find the value of $\frac{b+c}{2c-3b}$ when $b = 3$, $c = 7$.

64. Find the numerical value of $\frac{2a+2}{a-3} + \frac{3a-9}{a-2}$ when $a = 4$.

65. Find the numerical value of $3a - \frac{1}{2}(3b - 7(c - d))$ when $a = 15$, $b = 2$, $c = 3$, $d = 5$.

66. Find the numerical value of $\frac{a^2 + b^2}{c} + \frac{c^2 - d^2}{d}$ when $a = 1$, $b = 2$, $c = 3$, $d = 4$.

67. Find the numerical value of $\frac{b^2}{a} + \frac{a^2}{b} - \frac{c^2}{a}$ when $a = 1$, $b = 4$, $c = 6$.

68. Find the numerical value of $6ab^2 + 10a^2b - bc^2$ when $a = 1$, $b = 9$, $c = 8$.

69. What is the difference between $4a$ and a^4 when $a = 2$?

70. What is the difference between a^3 and $8a^2$ when $a = 8$?

(Section XVIII.)

Solve the equations

71. $12(x - 3) - 3(2x - 1) + 5x = 22.$

72. $\dfrac{x}{3} + 12 = \dfrac{3x}{4} + 7.$

73. $\frac{1}{2}x - \frac{1}{4}x = 5\frac{1}{4} - \frac{3}{4}x.$

74. $6(x + 5) = 8(57 - x).$

75. $104 - 10(2x - 1) = 54.$

76. $\dfrac{3}{7x} - \dfrac{5}{14} = \dfrac{1}{2}.$

77. $\dfrac{7x + 4}{2} - \dfrac{8x - 2}{7} = 5(40 - 2x).$

78. $\dfrac{4x - 6}{3x - 4} = \dfrac{8x - 10}{6x - 7}.$

79. $\dfrac{60}{x + 2} = \dfrac{30}{x - 1}.$

80. $\dfrac{32}{3x - 4} = \dfrac{54}{5x - 6}.$

81. $\frac{1}{2}(5x - 1) - 6(22 - 3x) = 2x - 3.$

82. $\dfrac{34x - 56}{15} - \dfrac{7x - 3}{5} = \dfrac{7x - 5}{3} + 2\frac{1}{6}.$

83. $\dfrac{2x}{3} - \dfrac{x - 3}{6} = \dfrac{1}{7}(5x - 4).$

84. $\dfrac{x+1}{2}+\dfrac{2(x+2)}{3}=\dfrac{9(x-3)}{4}.$

85. $\dfrac{x-5}{3}+\dfrac{x}{2}=12-\dfrac{x-10}{3}.$

86. $\dfrac{5x-6}{9}-\dfrac{2x-13}{3}=\dfrac{x+7}{3}.$

(Sections XXII, XXIV.)

Find x and y in the following simultaneous equations:

87. $\left.\begin{aligned}3x-4y&=25\\5x-2y&=7\end{aligned}\right\}.$

88. $\left.\begin{aligned}\frac{x}{2}+\frac{y}{3}&=8\\\frac{x}{4}-\frac{y}{12}&=1\end{aligned}\right\}.$

89. $\left.\begin{aligned}2x-3&=y\\3y+4&=5x\end{aligned}\right\}.$

90. $\left.\begin{aligned}4x+9y&=12\\6x-3y&=7\end{aligned}\right\}.$

91. $\left.\begin{aligned}8x-7y&=12\\\frac{x-2y}{4}+\frac{2x-y}{3}&=1\end{aligned}\right\}.$

92. $$\left.\begin{aligned}\frac{x}{5}-\frac{y}{7}&=1\\ \frac{y}{2}-\frac{x}{3}&=2\end{aligned}\right\}.$$

93. $$\left.\begin{aligned}\frac{x}{8}+\frac{y}{12}&=1\\ \frac{x}{2}-\frac{y}{6}&=1\end{aligned}\right\}.$$

94. $$\left.\begin{aligned}x+\frac{y}{3}&=19\\ \frac{x}{3}-y&=7\end{aligned}\right\}.$$

Find x, y, and z:

95. $$\left.\begin{aligned}x+y&=z\\ x-y+z&=4\\ 5x+y+2&=20\end{aligned}\right\}.$$

(Sections XX, XXI, XXIII.)

96. Eight times a certain number added to 16 is equal to 16 times a number one less. Find it.

97. Find a number such that 45 times the number increased by 60 is equal to 500 diminished by 10 times the number.

98. What two consecutive numbers are such that $\frac{1}{4}$ of the larger added to $\frac{1}{3}$ of the smaller is equal to 9?

99. If $\frac{1}{11}$ of the larger of two consecutive numbers taken from $\frac{1}{8}$ of the smaller leaves 7, what are they?

100. Two persons have equal sums of money, but the first owes the second 60 dollars; when he has paid his debt the second has twice as much as the first. How much had each?

101. Divide 21 into two such parts that one of them shall contain the other 21 times exactly.

102. Divide the decimal fraction .07 into two other decimal fractions which differ from each other by .007.

103. Find the number which increased by 5 is contained the same number of times in 45 as the same number diminished by 5 is contained in 12.

104. A purse of eagles is divided among three persons, the first receiving half of them and one more, the second half of the remainder and one more, and the third 6. Find the number of eagles the purse contained.

105. A person possesses $5,000 of stock. Some yields 3 per cent., four times as much yields $3\frac{1}{2}$ per cent., and the rest 4 per cent. Find the amount of each kind of stock when his income is $176.

106. Divide a yard into two parts such that half of one part added to 22 inches may be double the other part.

107. Divide $120 among three persons so that the first may have three times as much as the second, and the third one-fourth as much as the first and second together.

108. Two coaches start at the same time from two places,

A and B, 150 miles apart, one travelling 5 miles an hour, the other $6\frac{1}{2}$ miles an hour. Where will they meet, and at what time?

109. A number is written with two digits whose difference is 7, and if the digits be reversed the number so formed will be $\frac{2}{9}$ of the former. Find the original number.

110. Divide 200 into two parts so that one of them shall be two-thirds of the other.

111. A is three times as old as B; twelve years ago he was eleven times as old. What are their ages?

112. A father has five sons, each of whom is three years older than his next younger brother, and the oldest is four times as old as the youngest. Find their respective ages.

113. Divide 60 into two parts so that the difference of their squares shall be 1,200.

114. Divide 30 into three parts so that the ratio of the first two shall be 1 : 2, and that of the last two 5 : 3.

115. If $6x - 3 : 4x - 5 :: 3x + 5 : 2x + 3$, find x.

116. Eight horses and five cows consume a stack of hay in 10 days, and three horses can eat it alone in 40 days. In how many days will one cow be able to eat it?

117. One-fourth of a ship belongs to A and one-fifth to B, and A's part is worth \$6,000 more than B's. What is the value of the ship?

118. A certain fraction becomes $\frac{1}{3}$ if 2 be added to its

numerator, and if 2 be added to its denominator it becomes $\frac{1}{4}$. What is the fraction?

119. If a certain number be multiplied by $7\frac{2}{3}$, the product is as much greater than 16 as the product of its multiplication by $2\frac{5}{6}$ is less than 110. What is the number?

120. The highest pyramid in Egypt is 25 feet higher than the steeple of St. Stephen's Church in Vienna, and the height of this last is $\frac{17}{18}$ of the height of the pyramid. How high is each?

121. "The clock has struck ——," called out the night-watchman. "What hour did it strike?" asked a passer-by. The watchman replied: "The half, the third, and the fourth of the hour struck is one greater than the hour." What hour did it strike?

122. Of a swarm of bees the fifth part lighted on a blooming Cadamba, and the third part on the blossoms of Silind'hri, three times as many as the difference between the first two numbers flew to the flower Cutaja, and the one bee remaining hovered in the air unable to choose between the aromatic fragrance of the Jasmin and the Pandanus. "Tell me, beautiful girl," said the Brahmin, "the number of bees."

123. A father died, and left to his two sons and his wife $30,000, with the conditions that the share of the elder brother should be to the share of the younger as 4 : 3; and the share of the mother should be $\frac{3}{8}$ of the amount left to both brothers. What was the share of each?

124. A farmer grazes a certain number of sheep and oxen in two fields. One contains 13 animals, but only half the sheep and one-fourth the oxen are in it. The other field contains 21 animals. How many of each sort had he in the fields?

125. Find two numbers in the ratio of 3 to 4, whose sum is to 1 as 30 added to the second is to 2.

126. If 8 times one number be taken from 7 times another number 3 remains; and 9 times the first added to 6 times the second is 96. What are the numbers?

127. A said to B: "If you give me 7 dollars of your money, then I shall have twice as much as will remain to you." B said to A: "If you give me 4 dollars, then I shall have twice as much as remains to you." How much had A and B, each?

128. The sum of two numbers is 40, and their quotient is 3. What are the numbers?

129. Find two consecutive numbers whose product diminished by 20 equals the square of the first.

130. The hour and minute hands of a clock are together at 12 o'clock, when will they be together again?

If x = number minute-spaces gone over by minute-hand before they are together again, then, $\frac{1}{12}x$ = number minute-spaces gone over by hour-hand before they are together again. And $\frac{11}{12}x$ = gain of minute-hand. And, since this gain must be 60 minute-spaces,

$$\therefore\ \frac{11}{12}x = 60.$$

(Sections XXV, XXVI.)

Find the squares of the following quantities:

131. (1), $x^2 - 3x + 4$. (2), $4a - 2b + 3c$. (3), $x - a + 2b - c$.

Find the square roots of the following quantities:

132. $49x^2 + 126ax + 81a^2$.

133. $121a^2 - 330ab + 225b^2$.

134. $400a^2x^2 - 200abx + 25b^2$.

135. $4a^2 - 12ab + 9b^2 + 20ac - 30bc + 25c^2$.

136. $x^4 - 4x^3 + 6x^2 - 4x + 1$.

(Sections XXVII–XXIX.)

Find the value of x in each of the following equations:

137. $(x + 3)^2 = 6x + 25$.

138. $\dfrac{3}{1 + x} + \dfrac{3}{1 - x} = 8$.

139. $\frac{1}{2}(9 - 2x^2) = \frac{3}{2} - \frac{1}{10}(7x^2 - 18)$.

140. $\dfrac{2 - x^2}{3} + \dfrac{3 - x^2}{4} + \dfrac{4 - x^2}{5} = \dfrac{x^2 - 5}{6} - \dfrac{3}{4}$.

141. $2x^2 - 12x + 16 = 160$.

142. $4x^2 - 32x + 40 = 76.$

143. $x^2 - x - 40 = 170.$

144. $3x^2 + 2x - 9 = 76.$

145. $x^2 - 2bx = a^2 - b^2.$

146. $16 - \frac{2x^2}{3} = \frac{4x}{5} + 7\frac{3}{5}.$

147. $x^2 - \frac{x}{3} = 14x + 10.$

148. There are three numbers in ratio $\frac{1}{2} : \frac{2}{3} : \frac{3}{4}$, the sum of whose squares is 724. Find them.

149. Find the number which added to its square gives 182.

150. There are two numbers one of which is $\frac{3}{4}$ of the other, and the difference of their squares is 63. Find them.

151. There is a rectangular bathing pool whose length exceeds its breadth by ten feet, and it contains 1,200 square feet. Find its length and breadth.

152. The difference of two numbers is 5, and their product is 1,800. Find them.

153. The product of two numbers is 126, and if one be increased by 2 and the other by 1, their product is 160. Find them.

154. Find two consecutive numbers whose product is 600.

(Sections XXX, XXXI.)

Find x and y in the following simultaneous equations :

155. $\left.\begin{aligned} x - y &= 15 \\ \frac{xy}{2} &= y^2 \end{aligned}\right\}.$

156. $\left.\begin{aligned} x + y : x - y &:: 13 : 5 \\ y^2 + x &= 25 \end{aligned}\right\}.$

157. $\left.\begin{aligned} y^2 - 10x &= 10y + 36 \\ x + 2y &= 36 \end{aligned}\right\}.$

158. $\left.\begin{aligned} 3x + 2y &= 20 \\ 2x^2 - y^2 &= 71 \end{aligned}\right\}.$

159. Reduce $\sqrt{45} - \sqrt{20} + \sqrt{50} + \sqrt{125} - \sqrt{180}$.

160. Square $1 + \sqrt{3}$.

161. Multiply $4 - \sqrt{3}$ by $4 + \sqrt{3}$.

162. Simplify $\sqrt{512a^3b^2c^2}$.

163. Simplify $\sqrt{\frac{5}{7}}$.

164. Simplify $\dfrac{8}{3 - \sqrt{5}}$.

165. Simplify $\dfrac{4}{\sqrt{5} - 1}$.

166. Multiply $\sqrt{3} + 2\sqrt{2}$ by $2\sqrt{3} + \sqrt{2}$.

Find x in the following equations :

167. $\sqrt{x} + \sqrt{x - 7} = 7$.

168. $\sqrt{5x+11} + \sqrt{5x-9} = 10.$

169. $\sqrt{x+1} + \sqrt{x-1} = \dfrac{2}{\sqrt{x+1}}.$

(Section XXXII.)

170. Compare the ratios

$$4 : 5 \text{ and } 15 : 16;$$

$$14 : 15 \text{ and } 22 : 23.$$

171. What is the ratio of a inches to c feet?

172. Find a fourth proportional to $\frac{2}{5}$, $\frac{3}{4}$, $\frac{5}{7}$.

173. What number is that to which if 2, 4, and 7 be severally added, the first sum is to the second sum as the second is to the third?

174. What two numbers are those whose difference, sum, and product are proportionate to the numbers 3, 5, and 20, respectively?

(Sections XXXIII, XXXIV.)

175. Find the sum of the progression 1, 7, 13, 19, etc., to 50 terms.

176. Find the sum of $\frac{2}{3}$, $\frac{13}{12}$, $1\frac{1}{2}$, etc., to 20 terms.

177. Insert five arithmetical means between 12 and 20.

178. The first and last of 30 numbers in arithmetical progression is $2\frac{1}{4}$ and $2\frac{1}{2}$. What are the intervening terms?

179. A certain number consists of three digits, which are in arithmetical progression; and the quotient of the number divided by the sum of its digits is 15; but if 396 be added to it, the digits will be inverted. What is the number?

Let $x =$ the middle digit, and $y =$ the common difference; then $x - y$, x and $x + y =$ the digits, respectively; and the number $= 100(x - y) + 10x + (x + y) = 111x - 99y$.

Hence, the simultaneous equations:

$$\frac{111x - 99y}{3x} = 15$$

$$111x - 99y + 396 = 100(x + y) + 10x + (x - y).$$

180. The population of a town increases in the ratio of $\frac{1}{10}$ annually; it is now 10,000. What will it be at the end of four years?

181. Find the geometric mean between $\frac{a + x}{a - x}$ and $a^2 - x^2$.

182. Insert two geometric means between 5 and $-\frac{5}{8}$.

183. Insert 3 geometric means between 12 and 972.

184. Find the value of the recurring decimal .81818181 as a decreasing geometrical progression.

185. A farmer sowed a bushel of wheat and used the whole produce, 15 bushels, for seed the second year, the produce of this second year for seed the third year, and the produce of this again for the fourth year. Supposing the increase to have been always in the same proportion to the seed sown, how many bushels of wheat did he harvest at the end of the fourth year?

SECTION XXXVI.

GENERAL REVIEW QUESTIONS.

I.

1. How are quantities represented in algebra? Why do we use letters?

2. What is the chief point of distinction between digits and letters as used in algebra?

3. What are the signs of addition and subtraction?

4. What are positive quantities? Negative quantities?

5. What are the signs of multiplication?

6. When may we omit them and still denote multiplication?

7. What is the difference between 345 and *abc* when $a = 3$, $b = 4$, $c = 5$?

8. What are the signs of division?

II.

1. What is a factor? What is a coefficient?

2. What is a power?

3. What is an index? What other name have we for the index of a power?

4. What is an algebraic expression?

5. What are the terms of an expression?

6. What is a monomial? a binomial? a trinomial? a polynomial? Give an example of each.

7. What are like terms? Unlike terms?

8. How do we simplify an algebraic expression?

9. What is the rule for addition?

10. What is the rule for subtraction?

11. Show the reason for the rule of the signs in subtraction.

12. How does algebraic addition differ from arithmetical? Illustrate by an example.

13. How does algebraic subtraction differ from arithmetical? Illustrate by an example.

III.

1. State and prove the rule of the signs in multiplication.

2. How do we multiply monomials together?

3. Give the rule for multiplying a polynomial by a monomial.

4. Give the rule for multiplying a polynomial by a polynomial.

5. State and prove the rule of signs in division.

6. What is the rule for dividing a monomial by a monomial? A polynomial by a monomial?

7. Give the rule for the division of polynomials.

8. What is meant by arranging a polynomial with reference to a certain letter? Give an example.

IV.

1. What is the square of the sum of two quantities equal to?

2. Express this by a formula, when a and b are the quantities.

3. What is the square of the difference of two quantities equal to?

4. Express this by a formula.

5. What is the product of the sum and difference of two quantities equal to?

6. Express this by a formula.

7. What are brackets?

8. Give the rules for removing brackets.

9. Show by examples how to remove brackets with the plus sign before them; with the minus sign before them; with the sign of multiplication before them.

V.

1. What is the product of $x + a$ by $x + b$? Of $x - a$ by $x - b$? Of $x + a$ by $x - b$?

2. What are the factors of $x^2 + 5x + 6$? Of $x^2 - 13x + 40$? Of $x^2 + 5x - 6$? Of $x^2 - x - 30$?

3. What are the factors of $x^2 + 4x + 4$? Of $x^2 - 6x + 9$? Of $4x^2 - 9a^2$?

4. What is the least common multiple of two algebraic expressions?

5. Give the rule for finding this L. C. M.

6. What is the G. C. D. of two algebraic expressions.

7. Give the rule for finding it.

VI.

1. Do the rules for the operations of reduction, addition, multiplication, etc., on algebraic fractions differ from those in the arithmetic for common fractions?

2. Give these rules.

3. What is meant by the numerical value of an algebraic expression? and how do we find it in any supposed case?

4. What is the difference between a^2 and $2a^2$, when $a = 2$?

5. What is the value of $a^2 - 2ab + b^2$, when $a = 6$ and $b = 5$?

VII.

1. What is an equation?

2. What is an identity?

3. What is a known, and what an unknown quantity?

4. What is a simple equation?

5. What is meant by the solution of an equation?

6. Enumerate the operations which may be performed

on an equation without destroying the equality of the two sides.

7. Give the rule for the solution of a simple equation with one unknown quantity.

8. When is the solution of an equation said to be verified.

9. Give the rule for solving problems by equations.

10. Give some examples of translation from common into algebraic language.

VIII.

1. What are simultaneous equations?

2. What are the different methods of eliminating one of the unknowns in simultaneous equations?

3. Explain these methods.

4. How do we proceed when we have three simultaneous equations with three unknown quantities?

IX.

1. What is meant by involution?

2. What is the rule for the square of the sum of three quantities? Give the square of $a + b + c$.

3. What is meant by evolution?

4. What is the square root of a number? The cube root? Fourth root?

5. What is the *Radical Sign?* How do we indicate

the square root, cube root, fourth root, etc., of a quantity?

6. What sign has the square root of a quantity? The fourth root?

7. Why has a negative quantity no square root?

X.

1. How do we find the square root of a monomial? of a fraction?

2. When is a trinomial a perfect or complete square?

3. How do we find the square root of a trinomial, when it is a perfect square?

4. What term added will render a binomial of the form $x^2 + px$, or $x^2 - px$, a perfect square? Illustrate by an example.

5. Give the rule for finding the square root of a polynomial.

XI.

1. What is a quadratic equation, or equation of the second degree?

2. What is a pure quadratic?

3. What is an affected quadratic?

4. How many values has the unknown in a quadratic equation?

5. How do we solve a pure quadratic?

6. Give the steps in the solution of an affected quadratic, and the reasons for them.

7. In the solution of simultaneous equations with two unknowns by quadratics, what is the usual mode of elimination?

XII.

1. What is a radical expression, or surd?

2. What is a radical of the second degree?

3. What is the coefficient of a radical?

4. What is the difference between $2\sqrt{x}$ and $2+\sqrt{x}$ when $x=81$?

5. What is the difference between $\sqrt{a+b}$ and $\sqrt{a}+b$, when $a=9$ and $b=16$?

6. What is the difference between $\sqrt{\frac{a}{b}}$ and $\frac{\sqrt{a}}{b}$ when $a=36$ and $b=9$?

7. How may we transfer the coefficient of a radical of the second degree as a factor under the radical sign without affecting the value of the expression? Give an example.

8. How may we transfer a square factor from under a radical sign of the second degree as a coefficient before it? Give an example.

9. How do we use this to simplify a radical of the second degree?

10. What are similar radicals of the second degree? How do we add or subtract them?

11. How do we multiply two radicals of the second degree? How do we divide them?

12. How do we simplify radical expressions of the form $\sqrt{\frac{a}{b}}$? Illustrate by a numerical example.

13. How do we simplify expressions of the form $\frac{c}{a+\sqrt{b}}$, or $\frac{c}{a-\sqrt{b}}$? Illustrate by an example.

14. How do we solve equations of the forms $\sqrt{x+b}=c$, $\sqrt{x+a}+\sqrt{x+b}=c$?

XIII.

1. What is ratio?

2. What is proportion?

3. What three important things are to be remembered in operating on ratios and proportions? (See Art. 117.)

4. How do we simplify a ratio?

5. What are the *extremes* and what the *means* of a proportion?

6. What equation holds between them?

7. When are three quantities said to be in proportion? How do we find the mean proportional between two quantities?

8. What is the single Rule of Three?

9. Give the leading proportions which may be obtained from the proportion $a : b :: c : d$.

XIV.

1. What is an arithmetical progression? Increasing? Decreasing? In what general forms may it be written?

2. How do we find any term of an arithmetical progression when the first term and common difference are given?

3. Give the rule for finding the sum of the terms of an arithmetical progression.

4. How do we insert a given number of arithmetical means between two numbers?

5. How do we find the arithmetical mean or average of two numbers?

XV.

1. What is a geometrical progression? Increasing? Decreasing? In what general form may it be written?

2. How do we find any term of a geometrical progression when the first term and ratio are given?

3. Give the rule for finding the sum of the terms.

4. Give the rule for finding the limit of the sum of the terms of a decreasing geometrical progression when the number of terms is infinite.

5. How may this rule be applied to find the limiting value of a circulating decimal? Illustrate by an example.

6. How do we find the geometric mean between two numbers?

7. How do we insert two geometric means between two numbers? How do we insert three?

ANSWERS.

Examples—2, page 12.

1. $12a$. **2.** $2x + 2y + 2z$. **3.** $2 + 5a$. **4.** $2a^2 + 2b^2$. **5.** $3x^4 - x^2 - 15x - 2$. **6.** $2a^4 + 2a^2b + 3ab^2 - b^4$. **7.** $19ay^2$.

Examples—4, page 14.

1. $b + x$. **2.** $6b - 2c$. **3.** $7a - 5c$. **4.** $a - x - 9b$. **5.** $4x^2$. **6.** $2mn + 4m - 7n$. **7.** $2a^2b + 5a^2c + 2c^2$. **8.** $\frac{1}{3}ab - bc + 2$. **9.** $a^3 - a + 5a^2x + 111ax^2 + 166x^3$.

Examples—5, page 15.

2. $36a^2 - 63ab$. **3.** $-60ab + 96a^2$. **4.** $40x^4 - 20ax^3 - 12a^2x^2$. **5.** $-4abx + 8acx - 12bdx$. **6.** $3a^2c + 6abc$. **7.** $-4ax^2y^3 - 10bxy^3 + 6cxy^2z$. **8.** $-21x^{10} + 14x^7 - 28x^4$.

Examples—6, page 17.

1. $ac + ay + cx + xy$. **2.** $5x^2 - 6x - 8$. **3.** $x^2 - x - 20$. **4.** $6x^2 - 17x + 12$. **5.** $x - 3x^2 + 2x^3$. **6.** $ac^3 - a^2bc - b^2c^2 + ab^3$. **7.** $110x^2 + 118ax + 24a^2$. **8.** $x + 3x^2 - xy - y - 2y^2$. **9.** $2a^2b - 3abc + 2a^2c - ab^2 + b^2c$. **10.** $x^4 - 1$. **11.** $25 + 6x^2 + x^4$. **12.** $x^2 + 8x + 16 - y^2$.

13. $9a^6x^4 - 4b^4y^2$. **14.** $2x^4 - 32$. **15.** $a^8x^8 + y^8$. **16.** $a^4 + 4b^4$. **17.** $a^3 + b^3 + c^3 - 3abc$. **18.** $1 + 6a^6 + 5a^6$. **19.** $x^4 - 10x^3 + 25x^2 - 81$. **20.** $x^3 - 6x^2 + 12x - 8$.

Examples—7, page 18.

1. 3. **2.** $-2b$. **3.** $-6y$. **4.** b^2. **5.** b. **7.** $8a - 5b$.

Examples—8, page 20.

1. $4x - 32$. **2.** 0. **3.** $125 - 5x^2$. **4.** $2cx$. **5.** -4.

Examples—9, page 23.

In these examples the pupils should give the answers orally.

Examples—11, page 25.

1. $2b + 3c - 4d$. **2.** $-a - bx + cy$. **3.** $-3ax + 4b - x^2$. **4.** $1 + 7ac - 2bc$.

Examples—12, page 26.

1. $x + 2$. **2.** $a + 1$. **3.** $3a + x$. **4.** $x - 8$. **5.** $x + y + z$. **6.** $a + b + c$. **7.** $a - 6$. **8.** $a^2 - ab + b^2$. **9.** $a - b$. **10.** $x^2 + x + 3$. **11.** $x - y - z$. **12.** $5a^2 + 3x^2$. **13.** $a - b - c$. **14.** $3x^2 - x + 2$. **15.** $3x^2 - 2abx - 2a^2b^2$. **18.** $3ab - 4x$. **19.** $4x^2 + 6ax + 9a^2$. **20.** $8a^3 + 12a^2 + 18a + 27$.

Examples—13, page 29.

These answers the pupil should give orally.

Examples—14, page 31.

1. $16a^4b$. **2.** $26ax^3$. **3.** $12c^5y^2$. **4.** $x+1$. **5.** $x-2$. **6.** $x+9$. **7.** $x-1$. **8.** $3x-4$. **9.** $x-4$.

Examples—15, page 33.

1. $24ab$. **2.** $2a^2b^2c$. **3.** $240a^4$. **4.** a^2bc^2. **5.** $120x^4y^2$. **6.** $126a^6$. **7.** $36a^3b^3$. **8.** $a(x^2-y^2)$. **9.** $6(a^2-b^2)$. **10.** x^3-4x^2+5x-2.

Examples—16, page 34.

1. $\frac{a}{b}, \frac{a}{d}$. **2.** $\frac{a-b}{b}, \frac{a-b}{b}$. **3.** $\frac{5b^2}{3ac}, \frac{1}{2bc}$. **4.** $\frac{x+y}{x-y}, \frac{4a-5b}{20c}$. **5.** $\frac{x-y}{x}, \frac{x-1}{x-2}$. **6.** $\frac{x-a}{x+a}, \frac{x-2}{x+2}$.

Examples—17, page 36.

1. $\frac{5x}{15}, \frac{3x}{15}, \frac{x}{15}$. **2.** $\frac{12ab^2x}{48cx^2}, \frac{4bc+4ac}{48cx^2}, \frac{3a^2x}{48cx^2}$.

3. $\frac{a^2-ax}{a-x}, \frac{2a^2}{a-x}$. **4.** $\frac{6}{6x}, \frac{3}{6x}, \frac{2}{6x}$. **5.** $\frac{10x+8}{18}, \frac{10x+17}{18}$. **6.** $\frac{65ax-13a}{52a^2}, \frac{2x+4}{52a^2}$.

Examples—18, page 38.

1. $\dfrac{x+y+z}{a}$. 2. $\dfrac{3a+b}{3x}$. 3. $\dfrac{7x}{9}$. 4. $\dfrac{13}{12a}$. 5. $\dfrac{5x}{3}$.

6. $\dfrac{8ab+7a}{24bc}$. 7. $\dfrac{3a+b}{4}$. 8. $\dfrac{13x-2}{24}$. 9. $\dfrac{24x-23}{6}$.

10. $\dfrac{b}{3}$. 11. $\dfrac{ac+b}{c}$. 12. $\dfrac{4}{3}$. 13. 0. 14. $\dfrac{15}{x-1}$.

15. $\dfrac{3a+20}{3}$. 16. $2\frac{1}{12}$.

Examples—19, page 40.

1. $\dfrac{x}{14}$. 2. $\dfrac{x}{9}$. 3. $\dfrac{3x+5}{6}$. 4. $\dfrac{3}{10}$. 5. $\dfrac{2c}{b}$.

6. $\dfrac{-2bc}{a^2-b^2}$. 7. 0. 8. $\dfrac{-2ax}{x^2+3x+2}$. 9. $\dfrac{10x+15}{x^2-4}$.

10. $\dfrac{-2xy-y^2}{x^2+xy}$. 11. $\dfrac{4ab}{a^2-b^2}$. 12. $\dfrac{10-2x}{25}$.

13. $\dfrac{a^2c-b-c}{a}$. 14. 0.

Examples—20, page 42.

1. $\dfrac{10a-10b}{a+b}$. 2. $\dfrac{6a-2b}{c}$. 3. $18x-30$. 4. $\dfrac{5a^2}{2}$.

5. $\dfrac{9-4x}{20}$. 6. $\dfrac{15ax}{7b^2c}$. 7. $\dfrac{1}{2mb}$. 8. $1-\dfrac{1}{x^2}$. 9. $\dfrac{1}{1-x^2}$.

10. $\dfrac{1}{1-a^2}$. 11. $\dfrac{(x-2)(x-2)}{(x-3)(x-5)} \times \dfrac{(x-2)(x-5)}{(x-1)(x-2)}$, or

cancelling, $\dfrac{(x-2)(x-2)}{(x-1)(x-3)}$.

Examples—21, page 43.

1. $\dfrac{bx}{cy}$. **2.** $\dfrac{ax}{b^2y}$. **3.** $\dfrac{3x}{28}$. **4.** $\dfrac{3}{4b}$. **5.** $\dfrac{25mb}{3}$.

6. $\dfrac{1-3c}{c}$. **7.** $-3ab^2$. **8.** $\dfrac{2ax^5y}{bc}$. **9.** $\dfrac{2x+1}{x-1}$. **10.** 15.

11. $\dfrac{a-b}{a}$. **12.** $\dfrac{(x+1)(x+3)}{(x-3)(x-2)} \times \dfrac{(x-3)(x+3)}{(x+1)(x+2)}$, or cancelling, $\dfrac{(x+3)^2}{x^2-4}$.

Examples—22, page 45.

1. -1 **2.** -3. **3.** -6. **4.** -8. **5.** -5.
6. 3. **7.** 6 ; 24 ; -17. **8.** 18 ; -105. **9.** $-112\frac{1}{2}$.
10. 41. **11.** 11 ; 19. **12.** $-1\frac{1}{6}$. **13.** 5 ; $23\frac{4}{5}$.
14. 80 ; 160. **15.** $-\frac{259}{432}$. **16.** 125 ; 8. **17.** 3 ; $9\frac{2}{3}$. **18.** $\frac{4}{35}$.

Examples—23, page 49.

1. 6 ; -4. **2.** 3 ; 8. **3.** 11 ; $6\frac{1}{4}$. **4.** 26.1 ; $8\frac{2}{3}$.
5. -1.

Examples—24, page 50.

1. $b-a$; $c+a$ **2.** $\dfrac{2a+b-c}{2}$; $\dfrac{c}{a}$.

3. $\dfrac{m-n-p}{a-b}$. **4.** $\dfrac{b-a-c}{2}$.

Examples—25, page 50.

1. 5 ; 4. **2.** -2 ; 6. **3.** 2. **4.** 5.

Examples—26, page 52.

1. 15, 60. **2.** 20. **3.** 144. **4.** 18. **5.** $2\frac{2}{3}$. **6.** $\frac{9a}{8}$. **7.** $51\frac{2}{3}$. **8.** 4. **9.** 2. **10.** 8. **11.** 1. **12.** 8, $20\frac{1}{2}$. **13.** 1. **14.** 18.

Examples—27, page 58.

1. 9. **2.** 20. **3.** 9, 24. **4.** 205, 615, 2460. **5.** 48. **6.** 60. **7.** 48. **8.** 14, 15, 16. **9.** 17, 18. **10.** 6, 18. **11.** 8, 20. **12.** 40. **13.** 80 years, 20 years. **14.** 18 women, 22 men, 50 children. **15.** $48.

Examples—28, page 61.

1. 24, 25, 26, 27. **2.** $2\frac{2}{11}$ hours. **3.** $13\frac{11}{45}$. **4.** 3 hours, 9 miles, and 12 miles. **5.** $2\frac{2}{59}$ days. **6.** 147 in each. **7.** 2 miles. **8.** $\frac{4}{7}$. **9.** 336,292. **10.** 28.

Examples—29, page 66.

1. $x=6$, $y=9$. **2.** $x=17$, $y=7$. **3.** $x=50$, $y=20$. **4.** $x=4$, $y=2$. **5.** $x=4$, $y=5$. **6.** $x=15\frac{1}{2}$, $y=-\frac{11}{3}$. **7.** $x=9$, $y=11$. **8.** $x=8$, $y=12$.

9. $x = 11$, $y = 7$. **10.** $x = 15$, $y = 2$. **11.** $x = 3$, $y = 7$. **12.** $x = 5$, $y = 3$. **13.** $x = 1800$, $y = 100$. **14.** $x = 8$, $y = 5$. **15.** $x = 19$, $y = 3$. **16.** $x = 7$, $y = 9$. **17.** $x = 50$, $y = 27\frac{1}{3}$. **18.** $x = 12$, $y = 4$.

Examples—30, page 68.

1. 24,48. **2.** 13. **3.** 18, 28. **4.** \$80, \$40. **5.** \$8,000, \$10,000. **6.** \$20,764$\frac{2}{7}$, \$15,235$\frac{5}{7}$. **7.** $\frac{4}{20}$. **8.** 24 cows, 36 horses. **9.** 24, 6. **10.** 32 in first class, 54 in second class.

Examples—31, page 71.

1. $x = 2$, $y = 3$, $z = 4$. **2.** $x = 5$, $y = -2$, $z = -3$. **3.** $x = 6$, $y = 2$, $z = 4$. **4.** $x = 5$, $y = 8$, $z = 9$. **5.** $x = 6$, $y = 18$, $z = 10$. **6.** $x = 1$, $y = 2$, $z = 3$. **7.** $x = \frac{1}{2}$, $y = 4$, $z = 2$.

Examples—32, page 76.

1. $27a^3b^6$. **2.** $-8a^6b^3c^3$. **3.** $\dfrac{a^4b^8}{16c^{12}}$. **4.** $64a^2b^2$.

5. $64a^4b^2c^6$. **6.** $\dfrac{9c^2x^2}{4a^2y^2}$. **7.** $\dfrac{9y^4}{4x^6}$. **8.** $a^2 + 4a + 4$.

9. $4b^2c^2 + 4bc + 1$. **10.** $9m^2 - 30mn + 25n^2$. **11.** $a^2c^2x^2 + 2acxy + y^2$. **12.** $\dfrac{a^2b^2}{9} + \dfrac{2abc}{3} + c^2$.

Examples—33, page 79.

1. $\pm 5ay$. 2. $\pm 10a^2xy$. 3. $\pm 7ab^2$. 4. $\pm \dfrac{3ax}{2by}$.

5. $\pm \dfrac{4a^2b}{7x^3y^3}$. 6. $\pm \dfrac{3xy^2}{2a}$. 7. $\pm \dfrac{5a}{4}$. 8. $\pm \frac{4}{5}$. 9. $\pm \frac{1}{3}$.

10. $\pm \frac{5}{4}$.

Examples—34, page 80.

1. $\pm (a + 1)$. 2. $\pm (x - 2)$. 3. $\pm (x + \frac{5}{2})$.

4. $\pm (3ab - x)$. 5. $\pm (4a - 3b)$. 6. $\pm (8x^2 - \frac{3}{2}b)$.

7. $\pm (a + \frac{1}{2})$. 8. $\pm (4x - 1)$. 9. $\pm (x - \frac{3}{2})$.

Examples—35, page 81.

1. $\pm (x + 3)$. 2. $\pm (x - 6)$. 3. $\pm (x - \frac{11}{2})$.

4. $\pm (x - \frac{1}{2})$. 5. $\pm (y + \frac{3}{2})$. 6. $\pm (x - \frac{p}{2})$.

7. $\pm (a - \frac{4}{3})$. 8. $\pm (x - 2a)$. 9. $\pm (y + \frac{2}{7})$.

10. $\pm (x - \frac{7}{10})$. 11. $\pm (x - \frac{9}{20})$. 12. $\pm (x + \frac{5}{4})$.

Examples—36, page 83.

1. $8a + 9b$. 2. $x^2 + 2ax + a^2$. 3. $x^2 - 2x - 2$.

4. $3a - 2b + c$. 5. $3a^2 - 2a + 1$. 6. $4a^2 - 2b + c^2$.

7. $3c^2 - a + 2$. 8. $a - b + 2c$. 9. $x^2 - x - 1$.

10. $\dfrac{x^2}{3} - \frac{1}{2}$. 11. $a^2 - 5a + 6$.

Examples—37, page 85.

1. $\pm\sqrt{\frac{9}{5}}$. **2.** $\pm\sqrt{\frac{24}{5}}$. **3.** $\pm\frac{75}{2}$. **4.** ± 2. **5.** ± 5. **6.** ± 6. **7.** $\pm\frac{2}{3}$. **8.** ± 8. **9.** ± 4. **10.** ± 2.

Examples—38, page 88.

1. 15 or 3. **2.** 4 or -1. **3.** 5 or -1. **4.** $-1\pm\sqrt{\frac{3}{2}}$. **5.** $3\pm\sqrt{-46}$. **6.** 7 or -6. **7.** $\frac{2}{5}$ or $\frac{1}{5}$. **8.** 10 or $-\frac{11}{2}$. **9.** $\frac{1}{10}$ or $-\frac{9}{10}$. **10.** -1 or $\frac{16}{9}$. **11.** $\frac{14}{11}$ or -1. **12.** $-\frac{3}{2}$ or 2. **13.** $-\frac{1}{9}$ or $+9$. **14.** -7 or $+6$. **15.** 2 or $-\frac{2}{5}$. **16.** 4 or $\frac{20}{3}$. **17.** 5 or 3.

Examples—39, page 90.

1. 28. **2.** 9. **3.** 9. **4.** 100. **5.** 9, 15. **6.** $\frac{1}{3}$ or $\frac{2}{3}$. **7.** 10, 5. **8.** 10, 11. **9.** 3. **10.** 25. In these answers the negative results are omitted.

Examples—40, page 98.

1. $\left.\begin{array}{l} x = 2 \text{ or } 3 \\ y = 3 \text{ or } 2 \end{array}\right\}$. **2.** $\left.\begin{array}{l} x = 6 \\ y = 5 \end{array}\right\}$. **3.** $\left.\begin{array}{l} x = 8 \text{ or } 6 \\ y = 6 \text{ or } 8 \end{array}\right\}$.

4. $\left.\begin{array}{l} x = 8 \text{ or } \frac{1}{2} \\ y = 3 \text{ or } -4\frac{1}{2} \end{array}\right\}$. **5.** $\left.\begin{array}{l} x = \pm 12 \\ y = \pm 36 \end{array}\right\}$. **6.** $\left.\begin{array}{l} x = \pm 24 \\ y = \pm 20 \end{array}\right\}$.

7. $\left.\begin{array}{l} x = \pm 3 \\ y = \pm 2 \end{array}\right\}$. **8.** $\left.\begin{array}{l} x = 11 \\ y = 8 \end{array}\right\}$. **9.** $\left.\begin{array}{l} x = 4 \text{ or } -\frac{71}{13} \\ y = 5 \text{ or } \frac{106}{13} \end{array}\right\}$.

10. $\left.\begin{array}{l} x = 5 \text{ or } -9 \\ y = 2 \text{ or } -12 \end{array}\right\}.$ 11. $\left.\begin{array}{l} x = 5 \text{ or } 6\frac{3}{11} \\ y = 2 \text{ or } \frac{8}{11} \end{array}\right\}.$

Examples—41, page 98.

1. $2abc^2\sqrt{7a}$. 2. $x\sqrt{x-a}$. 3. $3\sqrt{3}$, $2\sqrt{3}$, $20\sqrt{3}$.
4. $2\sqrt{2}$, $5\sqrt{2}$, $6\sqrt{2}$. 5. $47\sqrt{2}$. 6. $12 - 10\sqrt{6}$.
7. $3 + \sqrt{3}$. 8. $\sqrt{a^2 - b^2}$. 9. 2. 10. $7 - 4\sqrt{3}$.
11. $\frac{1}{7}\sqrt{21}$. 12. $\frac{3}{8}\sqrt{2}$. 13. $\frac{2}{15}\sqrt{15}$. 14. $3\sqrt{2} + 3$.
15. $\sqrt{3} - 1$. 16. $\frac{1}{3}(\sqrt{5} + \sqrt{2})$. 17. 0.

Examples—42, page 100.

1. $b^2 - 2bc + c^2 - a$. 2. 144. 3. 36. 4. 5. 5. 25.
6. 3. 7. 5. 8. 2. 9. 16.

Examples—43, page 102.

1. $\frac{1}{7}$; $\frac{a}{n}$. 2. $\frac{1}{4a^2}$; $\frac{cx}{3}$. 3. $\frac{a}{b}$; $\frac{5c}{4a}$. 4. $\frac{bx}{5a}$; $\frac{a}{3c}$.
5. $\frac{a+b}{2x}$; $\frac{2a+b}{c}$. 6. $1 - a$; $a + x$. 7. $\frac{3b}{4a}$; $\frac{20m}{7n}$.
8. 17; 18.

Examples—44, page 105.

2. 7. 3. $x : y :: 5 : 4$. 4. 11. 5. $4bm$. 6. 8 and 12. 8. 24, 30. 9. 11. 10. $4 + x^2$. 11. 2 or -12. 12. 21, 22, 23, 24.

Examples—45, page 110.

1. (1.) 55, 111. (2.) 44, 86. (3.) $13\frac{1}{2}$, $27\frac{1}{2}$. (4.) 135, 65.

2. (1.) 590. (2.) 1240. (3.) $236\frac{2}{3}$. **3.** 300. **4.** \$81. **5.** 1,700 yards. **8.** $\frac{4}{15}$. **9.** a. **10.** $10\frac{1}{3}$, $10\frac{2}{3}$. **11.** Com. dif. 2. **12.** Com. dif. $\frac{1}{63}$, series, $3\frac{21}{63}$, $3\frac{22}{63}$, etc. **13.** $6433\frac{1}{3}$.

Examples—46, page 115.

1. (1.) 3. (2). 2. (3.) $\frac{3}{2}$. (4.) .1. (5.) 2. (6.) $3x$.

2. 25. **3.** 18, 54. **4.** $\frac{1}{3}$, $\frac{1}{9}$. **5.** The arithmetical. **6.** 25924. **7.** 2 dollars and 55 cents. **8.** $\frac{1}{3}$. **9.** $\frac{5}{9}$. **10.** $\frac{21}{55}$. **11.** 24, 48, 96.

Miscellaneous Examples, page 117.

1. $-3a^2x+3ax^2-2x^3$. **2.** $-6a^2b-4a^2$. **3.** $4a-7c$. **4.** $4a^2-9b^2$. **5.** $9a^2-b^2$. **6.** $a^2+2ab+b^2-c^2-2cd-d^2$. **7.** $a^4+a^2b^2+b^4$. **8.** $25a^2+1000ab+10000b^2$, $196a^2-308ab+121b^2$. **9.** x. **10.** $6ax-4x$. **11.** $27x^3+9x^2y+3xy^2+y^3$. **12.** a^3-2a^2+2a-1. **13.** $x^2+2ax+3a^2$. **14.** $3a-4x+b$. **15.** $4x^3-6cx^2+2c^2x$. **16.** $8+6x$. **17.** $1-x^4$. **18.** x. **19.** $(a-b)(a+b)(a^2+b^2)$. **20.** $(x-5)(x-6)$. **21.** $(x-6)$

$(x + 5)$. **22.** $(2m + 3n)(2m - 3n)$. **23.** $(4ax + 5y)(4ax - 5y)$. **24.** $(x - 7)(x - 2)$. **25.** $(x - 7)(x + 1)$. **26.** $b + 3$.

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27. ax. **28.** $a - b$. **29.** $x + 3$. **30.** $a + 1$. **31.** $3a - 2$. **32.** $63x^7$. **33.** $1260a^2x^3$. **34.** $4420a^2b^2c^2$. **35.** $18(x^2 + 3x + 2)$. **36.** $90(x^2 - 1)$. **37.** $36(x^2 - a^2)$. **38.** $a^3 + 7a^2 - a - 7$.

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39. $\frac{7ab}{2c^3}$. **40.** $\frac{2bc}{4a}$, $a - x$, $\frac{x + 2}{x + 1}$. **41.** $\frac{51a}{20x}$.

42. $\frac{x + a}{a - b}$. **43.** $\frac{2ac}{a^2 - x^2}$. **44.** $\frac{2}{a - b}$. **45.** $\frac{4x}{x^2 - 1}$.

46. $\frac{15}{x - 2}$. **47.** $\frac{6}{x^2 - 1}$. **48.** $\frac{4a}{a^2 - 1}$. **49.** $\frac{2a + 1}{15}$.

50. $\frac{1 - a}{1 + a}$ and $\frac{1}{1 - a}$.

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51. $\frac{1}{a^2 - 1}$. **52.** $4a - 2$. **53.** $\frac{x^2 - 1}{x^2 - 9}$. **54.** 1.

55. $\frac{1}{a^2 + a - 2}$. **56.** $\frac{6ac}{x}$. **57.** $\frac{1}{a + b}$. **58.** $\frac{a}{1 - a}$.

59. $\frac{1}{a}$. **60.** $\frac{a^2 + a}{2a + 1}$. **61.** $\frac{x - 2}{x - 3}$. **62.** 5. **63.** 2.

64. $11\frac{1}{2}$. **65.** 35. **66.** $-\frac{1}{12}$. **67.** $-19\frac{3}{4}$. **68.** 0.

69. 8. **70.** 0.

71. 5. **72.** 12. **73.** $5\frac{1}{4}$. **74.** $34\frac{5}{7}$. **75.** 3.

76. 1. **77.** 16. **78.** 1. **79.** 4. **80.** 12. **81.** 7.

82. $-2\frac{21}{44}$. **83.** 5. **84.** $7\frac{12}{13}$. **85.** $14\frac{4}{7}$. **86.** 3.

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87. $\left.\begin{array}{l} x = -1\frac{4}{7} \\ y = -7\frac{3}{7} \end{array}\right\}.$ **88.** $\left.\begin{array}{l} x = 8 \\ y = 12 \end{array}\right\}.$ **89.** $\left.\begin{array}{l} x = 5 \\ y = 7 \end{array}\right\}.$

90. $\left.\begin{array}{l} x = \frac{3}{2} \\ y = \frac{2}{3} \end{array}\right\}.$ **91.** $\left.\begin{array}{l} x = 12 \\ y = 12 \end{array}\right\}.$ **92.** $\left.\begin{array}{l} x = 15 \\ y = 14 \end{array}\right\}.$

93. $\left.\begin{array}{l} x = 4 \\ y = 6 \end{array}\right\}.$ **94.** $\left.\begin{array}{l} x = 19\frac{1}{5} \\ y = -\frac{3}{5} \end{array}\right\}.$ **95.** $\left.\begin{array}{l} x = 2 \\ y = 8 \\ z = 10 \end{array}\right\}.$

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96. 4. **97.** 8. **98.** 15, 16. **99.** 65, 66.

100. 180 dollars. **101.** $\frac{21}{22}$, $20\frac{1}{22}$. **102.** .0315, .0385.

103. $8\frac{7}{11}$. **104.** 30. **105.** \$800, \$3,200, \$1,000.

106. 20 inches, 16 inches. **107.** \$24, \$72, \$24.

108. $65\frac{5}{23}$ miles from A, $13\frac{1}{23}$ hours after starting.

109. 81. **110.** 80, 120. **111.** 45 years of age, and 15 years of age. **112.** 4, 7, 10, 13, 16 years of age respectively. **113.** 40, 20. **114.** $7\frac{1}{7}$, $14\frac{2}{7}$, $8\frac{4}{7}$. **115.** $-2\frac{2}{7}$.

116. 150. **117.** \$120,000. **118.** $\frac{8}{30}$. **119.** 12.

120. 425 feet, 450 feet. **121.** 12 o'clock. **122.** 15. **123.** Eldest son, \$12,467.52; youngest son, \$9,350.64; the widow, \$8,181.84. **124.** 18 sheep, 16 oxen. **125.** 9, 12. **126.** $6\frac{24}{111}$, $7\frac{44}{111}$. **127.** A, \$15, and B, \$18. **128.** 10, 30. **129.** 20, 21. **130.** $5\frac{5}{11}$ minutes past one o'clock.

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131. (1) $x^4 - 6x^3 + 17x^2 - 24x + 16$; (2) $16a^2 - 16ab + 24ac + 4b^2 - 12bc + 9c^2$; (3) $x^2 - 2ax + 4bx - 2cx + a^2 - 4ab + 4bc^2 + 2ac - 4bc + c^2$. **132.** $7x + 9a$. **133.** $11a - 15b$. **134.** $20ax - 5b$. **135.** $2a - 3b + 5c$. **136.** $x^2 - 2x + 1$.

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137. ± 4. **138.** $\pm \frac{1}{2}$. **139.** ± 2. **140.** ± 2. **141**, 12, or -6. **142.** 9 or -1. **143.** 15 or -14. **144.** 5 or $-5\frac{2}{3}$. **145.** $b \pm a$. **146.** 3 or $-\frac{21}{5}$. **147.** 15 or $-\frac{2}{3}$. **148.** 12, 16, 18. **149.** 13 or -14. **150.** 12, 9. **151.** 30, 40. **152.** 40, 45. **153.** 14 and 9, or 18 and 7. **154.** 24, 25.

155. $\left.\begin{array}{l} x = 18 \text{ or } \frac{25}{2} \\ y = 3 \text{ or } -\frac{5}{2} \end{array}\right\}.$ **156.** $\left.\begin{array}{l} x = 9 \text{ or } -14\frac{1}{16} \\ y = 4 \text{ or } -6\frac{1}{4} \end{array}\right\}.$

157. $\left.\begin{array}{l} x = 2 \text{ or } 78 \\ y = 14 \text{ or } -24 \end{array}\right\}.$ **158.** $\left.\begin{array}{l} x = 6 \text{ or } 114 \\ y = 1 \text{ or } -161 \end{array}\right\}.$

159. $5\sqrt{2}.$ **160.** $4 + 2\sqrt{3}.$ **161.** 13.

162. $16abc\sqrt{2a}.$ **163.** $\frac{1}{7}\sqrt{35}.$ **164.** $2(3 + \sqrt{5}).$

165. $\sqrt{5} + 1.$ **166.** $10 + 5\sqrt{6}.$ **167.** 16. **168.** 5.

169. 1.

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170. $15 : 16$ greater by $\frac{11}{80}$; $22 : 23$ greater by $\frac{8}{345}$.

171. $\frac{a}{12c}.$ **172.** $\frac{75}{56}.$ **173.** 2. **174.** 5 20.

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175. 7400. **176.** $92\frac{1}{2}$. **177.** Common difference $\frac{4}{3}$. Series 12, $13\frac{1}{3}$, etc., ...20. **178.** Common difference $\frac{1}{116}$. Series $2\frac{1}{4}$, $2\frac{30}{116}$, $2\frac{31}{116}$, etc., ...$2\frac{1}{2}$. **179.** 135. **180.** 14641. **181.** $a + x$. **182.** $-\frac{5}{2}$, $\frac{5}{4}$. **183.** 36, 108, 324. **184.** $\frac{81}{\text{[illegible]}}$. **185.** 50625.

Venable, C.S.
Easy algebra.

NAME	DATE	NAME	DATE